Moral hazard premium:
Valuation of moral hazard under diffusive and jump risks

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Abstract

We provide an equilibrium asset pricing formula under moral hazard on the assumption of a power utility function and an endogenous riskless rate. Moral hazard is defined here as a firm's change of measure that is incontractible. We show explicitly in closed form that moral hazard distorts asset prices by (1) moving the market price of diffusive risk in the opposite direction to an investor's marginal utility, (2) amplifying the market price of jump risk, and (3) stipulating a positive premium, called a "moral hazard premium" in this paper, on the riskless rate. Thus, the risk-free rate puzzle, which was explored first by Weil (1989), is further exaggerated under moral hazard. Also, financial markets alleviate the allocation conflict caused by moral hazard.
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JEL Classification Codes: D51, D82, G12.

1 Introduction

Moral hazard is widely considered one of the most serious problems in finance. There exists a huge amount of literature on moral hazard in corporate finance (e.g. Tirole (2006, Section 3.2)). In a typical moral hazard model, the probability of success (i.e., probability measure) of a firm's project depends on its manager's hidden effort, which is incontractible. An investor then needs to give the manager an incentive to avoid opportunistic misbehavior. Therefore, the moral hazard distorts optimal risk sharing and allocation.

Still, such micro effects are not all the distortions caused in the financial world. When the effects are aggregated in markets, the moral hazard distorts the risk-averse investor's marginal utility (i.e., pricing kernel) as well, and thus affects the valuation of not only the firm but also all other financial assets in markets. A question is then raised: how much return on investment would an investor demand to be compensated for a loss caused by moral hazard? Surprisingly, however, there have not been many studies regarding the valuation of moral hazard in asset pricing and financial engineering (e.g. fixed-income investments, the term structure of interest rates, corporate risk management, and actuarial insurance).

A notable exception is Ou-Yang (2005), who studies equilibrium asset pricing in the presence of moral hazard on the assumption of an exponential utility function and an exogenous constant riskless rate. Still, Ou-Yang (2005, p.1283) himself discusses the importance of incorporating more general utility functions such as the power utility function. In addition, the exogeneity assumption of the riskless rate seems to have limited its applicability to financial practices.

The objective of this paper is to provide an explicit asset pricing formula under two types of risks, i.e., regular (diffusive) and rare-event (negative jump) risks, in the presence of moral hazard on the assumption of a power utility function and an endogenous riskless rate. Specifically, we incorporate moral hazard into a general-equilibrium exchange economy (or, a consumption-based capital asset pricing model (C-CAPM)[1]) under the two types of risks in continuous time. We solve the problem of the investor's optimal consumption/wealth allocation in financial markets when the process of production is subject to a firm manager's moral hazard. We then explicitly obtain an equilibrium state price,[2] which can be used for pricing any kind of security or loan in the presence of moral hazard.

The more specific model setup is as follows. There exist two players: the representative investor and a representative firm (i.e., the firm manager) over a finite time horizon $[0, T]$. Both players rank their consumption based on time-separable utility with common time preference. The investor's instantaneous utility function is of a constant relative risk aversion (CRRA) type, while the firm's is of a logarithm type. The firm produces a single, non-storable consumption good over time and shares it with the investor. No productive resources are utilized: the production is an endowment. Unlike in standard endowment economies, however, the firm can control the probability measure ex post by incurring effort costs. The effort to control it is unobservable to the investor and so is incontractible. Thus the endowment process is subject to moral hazard, which is defined here as the firm's ex-post, costly, strategic change of measure that is incontractible.

The investor then optimizes ex post her consumption/wealth with access to financial markets while designing ex ante an optimal incentive contract for the firm so that the manager does not

[1]See e.g. Lucas (1978), Breeden (1979), Cox et al. (1985), and Dana and Jeanblanc (2007, Ch.7).

[2]State price is defined as the price of a security that agrees to pay one unit of consumption on a particular time path and to pay zero unit of it on others. It is also known as state price density, pricing kernel, stochastic discount factor, and equivalent martingale measure.

expect better rewards from his opportunistic misbehavior. Thus the moral hazard and the financial markets are interconnected in equilibrium. We look at the macroeconomic effect of moral hazard in the financial markets.[3]

This paper consequently provides an explicit asset pricing formula under the moral hazard problem. We obtain equilibrium state prices, the processes of which are characterized by a triplet: a riskless rate and the market prices of diffusive and jump risks. We make clear the structural effect of moral hazard on the triplet. Specifically, owing to moral hazard, the market price of diffusive risk is distorted and moves in the opposite direction to the investor's marginal utility while the market price of jump risk is amplified. In addition, a positive premium, called a "moral hazard premium" in this paper, is stipulated on a riskless rate in equilibrium, as compared to the case with no moral hazard. It is because (1) the diffusive-risk premium is reduced, as the distortion of the probability measure due to moral hazard works as a hedge against the diffusive risk and (2) the investor demands compensation for a loss due to aggravated jump risk. In contrast to Ou-Yang (2005), the premium and the two market prices of risks are time-varying owing to the optimal dynamics of the firm's effort. The result implies that the risk-free rate puzzle, which was explored first by Weil (1989), is further exaggerated in the presence of moral hazard.

In addition, we draw corporate-finance implications. We show that, as compared to standard corporate-finance models, the markets alleviate the allocation conflict caused by moral hazard. In the standard corporate-finance models, the entire effect of moral hazard is absorbed only in distortion of allocations of consumption goods, because the investor has no access to financial markets. In contrast, in our model, the effect is absorbed in distortion of prices as well in the markets. Thus it is divided into the two parts: the price effect and the allocation effect. The markets play the role of relieving the allocation conflict via the price effect.

In relationships to previous literatures, this paper is not a generalization of Ou-Yang (2005), but rather, is complementary to it. Mathematically, the investor's optimization problem is subject to two constraints on states: an incentive constraint provoked by the moral hazard and the firm's participation constraint. As Yong and Zhou (1999, p.155) point out, it is hard, in general, to solve stochastic control problems with state constraints. Our model is no exception. Some approximations are necessary for obtaining explicit solutions. In particular, as compared to Ou-Yang (2005),

[3]We could extend this current model into a model with heterogeneous firms. For example, consider the simple case where one independent shock is associated with one individual firm. We could then examine a firm-specific effect of moral hazard.

our analytical difficulty is increased by generalizing the utility function as a power type function and endogenizing the riskless rate. In order to deal with the increased difficulty, we confine our attention only to a smaller set of contract forms. Specifically, we obtain closed-form solutions by restricting the contract form to a stationary linear contract, in that it is linear (i.e., proportional to the production) with a constant (i.e, time-independent) rate of change.

Despite such restriction, our explicit results are meaningful. We can argue that the linearity assumption of the contract is not restrictive when finding optimal solutions, because the entire system of equations in this model is linear.[4] On the other hand, however, the stationarity assumption is restrictive because the time horizon is finite. To investigate the effect of non-stationarity, we perturb the result obtained under the stationary linear contract, by using the Taylor series expansion. We then find that the non-stationarity attribute distorts the moral hazard premium while not changing the market prices of diffusive and jump risks. In this manner, our explicit solution is useful as a benchmark for numerical analyses when explicit solutions are not obtained in more general environments.

Our paper is also related to Cvitanić et al. (2009) and Cvitanić and Zhang (2007) in the continuous-time moral hazard literature.[5] As with their papers, we study optimal contracting on the assumption of more general utility functions than the exponential one.[6] But, our paper extends their principal-agent models into a general-equilibrium model where the investor has access to the financial markets.[7] In addition, we have several other departures from their papers. To draw asset-pricing implications over time, we assume that the consumption takes place throughout the time period, whereas they assume that the players consume only at the end of the period. We also assume that Brownian motions represent regular risk and Poisson processes represent rare-event risk, while they assume only a Brownian motion.

Furthermore, our model assumes that the firm directly controls the probability measure, in the spirit of standard discrete-time moral hazard models (e.g., Tirole (2007, Section 3.2)). On the other hand, Cvitanić and Zhang (2007) assume that the agent controls the drift rate, like most of

[4] As we will discuss in Section 4.2 below, if the firm's utility takes on types of forms other than a logarithm one, including the power type, then the system of the equations would not be linear anymore. The linearity assumption would then be restrictive.

[5] E.g., Holmström and Milgrom (1987), Schättler and Sung (1993), Sannikov (2008). For an extensive survey of the literature, see the book of Cvitanić and Zhang (2013).

[6] Cvitanić and Zhang (2007) examine the problem of adverse selection as well.

[7] Ou-Yang (2005) extends a principal-agent model with an exponential utility function (e.g., Holmström and Milgrom (1987) and Schättler and Sung (1993)) to an asset-pricing model.

the continuous-time moral hazard literature. In Cvitanić and Zhang (2007), due to the tractability of measure change via the Girsanov theorem, the agent's optimization problem is written in the weak formulation. However, in the weak formulation, not only the drift rate μ but also the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ and the Brownian motion B are controlled such that μ is adapted to the information set $\mathbb{F}$, which is endogenously controlled and is not necessarily the augmentation of the filtration generated by B.[8] It means that the agent's control is adapted to the information set that is generated by a history of production, neither a history of his own observable true shocks nor of the controls. In other words, when controlling the drift rate, the agent may continue to forget how he has controlled it until then, although he can observe the true shocks and the controls. That looks irrelevant to modeling the moral hazard problem.[9] To avoid the irrelevance, we assume that the agent controls the probability measure, rather than the drift rate. Also, we formulate the effort cost as relative entropy, which is a measure of statistical discrimination (i.e., a "distance") between the original measure and the controlled probability measure.[10] Recently, relative entropy has often been used to represent information costs in economics and finance.[11] Our paper is in line with that literature.

The rest of this paper is organized as follows. Next section defines the environment of our model. Section 3 gives a formal representation of the firm's and the investor's optimization. Section 4 defines market equilibrium and characterizes it explicitly. Section 5 examines equilibrium asset prices and draws asset-pricing implications. Final section concludes.

[8]As to the definitions of strong and weak formulations of stochastic control problems, see Yong and Zhou (1999).

[9]In most of the continuous-time moral hazard literature, the investor's optimization problem is written in the strong formulation, in which his information set is generated by histories of his efforts as well as true shocks. When the different formulations are used for the two players, however, the firm's optimal control may not be in the investor's admissible control set due to the different requirements on the measurability of the firm's controls (for more details, see e.g. Cvitanić and Zhang (2007, Remark 2.3)). Therefore, our paper and Cvitanić and Zhang (2007) solve the two players' optimization problems in the weak formulation commonly. On the other hand, Nakamura and Takaoka (2013) and Nakamura (2015) solve them in the strong formulation commonly and show that the two formulations are equivalent in equilibrium under some assumptions,

[10]See e.g. Cover and Thomas (2006, p.18) in statistics.

[11]See e.g. Hansen and Sargent (2007), Hansen et al. (2006) and Sims (2003) in economics. Also, in mathematical finance, Delbaen et al. (2002) use it as a penalty in hedging contingent claims.

2 Environment

2.1 Players and filtered probability space

We consider a dynamic stochastic economy with two players: a representative firm manager (simply called a firm) and a representative investor on a time interval $[0, T]$ for a finite time $T > 0$. The firm and the investor are indexed by player 1 and player 2, respectively. For convenience, we will use female pronouns for the investor, and male ones for the firm.

Define a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}(t)\}_{0\le t\le T}, \mathbb{P})$. $\{B_1(t), \cdots, B_n(t)\}_{0\le t\le T}$ are n independent one-dimensional standard $\mathbb{F}$-Brownian motions on the probability space, i.e., for any t, s satisfying $0 \le t \le s$, $B_j(s) - B_j(t)$ is independent of $\mathcal{F}(t)$ and $B_j(0) = 0$ for $j = 1, \cdots, n$. $\{N_1(t), \cdots, N_m(t)\}_{0\le t\le T}$ are m independent Poisson processes, each of which is characterized by its intensity $\lambda_i > 0$ for $i = 1, \cdots, m$. Let the compensated Poisson process be denoted by $M_i(t) := N_i(t) - \lambda_i t$, which is a $\mathbb{P}$-martingale. The Poisson processes are independent of $\{B_j(t); j = 1, \cdots, n\}_{0\le t\le T}$ as well. The filtration $\mathbb{F}$ is generated by $\{B_j(t); j = 1, \cdots, n\}_{0\le t\le T}$ and $\{N_i(t); i = 1, \cdots, m\}_{0\le t\le T}$. For notational convenience, we may also write an n dimensional process $B(t) := (B_1(t), \cdots, B_n(t))^\top$, an m dimensional process $N(t) := (N_1(t), \cdots, N_m(t))^\top$, and an m dimensional process $M(t) := (M_1(t), \cdots, M_m(t))^\top$. We may also call B diffusive risk and N jump risk.

Define a measure $\mathbb{Q}$ that is absolutely continuous w.r.t. $\mathbb{P}$, written as $\mathbb{Q} \ll \mathbb{P}$, i.e., $\mathbb{P}(A) = 0$ implies $\mathbb{Q}(A) = 0$ for $A \in \mathcal{F}$. Define also the Radon-Nikodym derivative process as

$$Z(t) := \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\bigg|_{\mathcal{F}(t)} = \mathbb{E}_t^{\mathbb{P}}[\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}].$$

By the Martingale Representation Theorem (cf. Theorem 5.43 of Medvegyev (2007)), there exist $\mathbb{F}$-predicable processes θ_j and $\alpha_i \ge -1$, where $\int_0^T (\theta_j(t))^2 \,\mathrm{d}t < \infty$ and $\int_0^T \alpha_i(t)\,\mathrm{d}t < \infty$, for all $i = 1, \cdots, m$ and all $j = 1, \cdots, n$ such that

$$\mathrm{d}Z(t) = Z(t_-)\Big\{\sum_{j=1}^n \theta_j(t)\,\mathrm{d}B_j(t) + \sum_{i=1}^m \alpha_i(t)\,\mathrm{d}M_i(t)\Big\}. \tag{2.1}$$

Once, due to a jump, $Z(\tau) = 0$ at some time τ, $Z(t) = 0$ for $t \ge \tau$. For each $j = 1, \cdots, n$,

$$\widetilde{B}_j(t) := B_j(t) - \int_0^t \theta_j(s)\,\mathrm{d}s \tag{2.2}$$

is a $\mathbb{Q}$-Brownian motion, and for each $i = 1, \cdots, m$,

$$\widetilde{M}_i(t) := N_i(t) - \int_0^t \widetilde{\lambda}_i(s)\,\mathrm{d}s \tag{2.3}$$

is a $\mathbb{Q}$-(local) martingale where $\widetilde{\lambda}_i(s) := \lambda_i\{\alpha_i(s) + 1\}$ (cf. Theorem 41 of Protter (2010, Ch.III)). Note that $\widetilde{B}_j(t)$ and $\widetilde{M}_i(t)$ (or $N_i(t)$) are uncorrelated instantaneously for any i, j, i.e., the quadratic variations $\mathrm{d}\widetilde{B}_j(t)\cdot\mathrm{d}N_i(t) = 0$ and $\mathrm{d}\widetilde{B}_j(t)\cdot\mathrm{d}\widetilde{M}_i(t) = 0$ for any i, j, but are not necessarily independent under $\mathbb{Q}$ while $B_j(t)$ and $M_i(t)$ (or $N_i(t)$) are independent under $\mathbb{P}$ for any i, j.

Applying Itô's formula to Eq.(2.1) (see e.g. T3 Theorem of Brémaud (1981, p.166) and Theorem 11.6.9 of Shreve (2004, p.503)),

$$\begin{aligned} Z(t) \;=\; & \prod_{j=1}^{n} \exp\left\{\int_0^t \theta_j(s)\,\mathrm{d}B_j(s) - \frac{1}{2}\int_0^t (\theta_j(s))^2\,\mathrm{d}s\right\}\cdot \\ & \prod_{i=1}^{m} \exp\left\{\sum_{0<s\le t} \log\left(\frac{\widetilde{\lambda}_i(s)}{\lambda_i}\right)\Delta N_i(s) + \int_0^t (\lambda_i - \widetilde{\lambda}_i(s))\,\mathrm{d}s\right\}. \end{aligned} \tag{2.4}$$

2.2 Production

The firm produces a single non-storable consumption good over time, denoted by X, which is characterized by the following stochastic differential equation (SDE):

$$\mathrm{d}X(t) = X(t_-)\,\mathrm{d}G(t) := X(t_-)\Big(\mu^G\,\mathrm{d}t + \sum_{j=1}^{n}\sigma_j^G\,\mathrm{d}B_j(t) + \sum_{i=1}^{m} z_i^G\,\mathrm{d}M_i(t)\Big), \quad X(0) = x_0 > 0 \tag{2.5}$$

where $\mu^G, \sigma_j^G, z_i^G\,\forall i, j$ are constants, $\sigma_j^G > 0\,\forall j$, $-1 < z_i^G < 0\,\forall i$, and $z_{i_1}^G \neq z_{i_2}^G$ if $i_1 \neq i_2$. No productive resources are utilized: the production is an endowment, which stands for real gross domestic product (GDP). In financial terms, $\{B_j;\, j = 1, \cdots, n\}$ stands for regular risk and $\{N_i;\, i = 1, \cdots, m\}$ stands for rare-event risk. For each $i = 1, \cdots, m$, z_i^G stands for the jump size of N_i, and $-1 < z_i^G < 0$ means that the jump causes a loss. $\sum_{i=1}^m z_i^G N_i$ can be interpreted as a mixed Poisson process with its intensity $\sum_{i=1}^m \lambda_i$, for which process $\frac{\lambda_i}{\sum_{i=1}^m \lambda_i}$ stands for the probability of having the jump size z_i^G when a jump occurs. We may also write an n dimensional row vector $\sigma^G := (\sigma_1^G, \cdots, \sigma_n^G)$ and an m dimensional row vector $z^G := (z_1^G, \cdots, z_m^G)$.

2.3 Moral hazard

We assume that the representative firm can control the probability measure in the whole economy so as to maximize his own expected payoff. More specifically, $\mathbb{P}$ is the original probability measure, that is, the measure when the firm would not control it – we may also call it the reference measure. The firm changes the probability measure from $\mathbb{P}$ into $\mathbb{Q}$ such that $\mathbb{Q} \ll \mathbb{P}$.[12] Assume that $\mathbb{P}$ is the public information, and that the investor knows the fact that $\mathbb{Q}$ is absolutely continuous w.r.t $\mathbb{P}$, but cannot observe $\mathbb{Q}$ directly, i.e., $\mathbb{Q}$ is the private information of the firm.

We also assume that the firm incurs a utility cost when controlling the probability measure. The cost is represented by relative entropy, denoted by $H(\mathbb{Q}||\,\mathbb{P})$,[13] which is defined as:

$$H(\mathbb{Q}\,||\,\mathbb{P}) := \mathbb{E}^{\mathbb{P}}\left[\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\left(\log\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)\mathbf{1}_{\{\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}>0\}}\right] = \mathbb{E}^{\mathbb{Q}}\left[\left(\log\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)\mathbf{1}_{\{\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}>0\}}\right] = \mathbb{E}^{\mathbb{Q}}\left[\log\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right].$$

Assume that $H(\mathbb{Q}\,||\,\mathbb{P}) < \infty$.[14] Roughly speaking, the relative entropy is a measure of the distance between the probability measures $\mathbb{P}$ and $\mathbb{Q}$.[15] From a statistical viewpoint, it represents a measure of the type-I error of rejecting the true probability measure $\mathbb{Q}$ and, instead, assuming $\mathbb{P}$ incorrectly. That is, it stands for the statistical inefficiency of assuming that the probability measure is $\mathbb{P}$ when the true measure is $\mathbb{Q}$. A low level of the relative entropy means that $\mathbb{Q}$ and $\mathbb{P}$ are not so distant as to significantly discriminate $\mathbb{P}$ against $\mathbb{Q}$. Thus, in this model, the relative entropy means how far the true probability measure $\mathbb{Q}$ is distorted from the reference measure $\mathbb{P}$. The effort cost impedes the firm's adopting the probability measure that is very far from $\mathbb{P}$.

In this framework, from Eq.(2.2), Eq.(2.3) and Eq.(2.4), we can characterize the relative entropy

[12] Under the absolute-continuity restriction, zero probability is necessarily assigned, under $\mathbb{Q}$, to the state to which zero probability is assigned under $\mathbb{P}$. In other words, this model does not look at the states that are supposed not to occur under the reference measure $\mathbb{P}$.

[13] This formulation of moral hazard is borrowed basically from a companion work of this paper, namely Misumi et al. (2013).

[14] Note that this finiteness assumption is imposed for removing the indeterminacy of the firm's optimal expected utility defined in Eq.(3.1) below.

[15] The relative entropy is always non-negative and is zero if and only if $\mathbb{Q} = \mathbb{P}$. Strictly speaking, it is not a true distance because neither the symmetry nor the triangle inequality is satisfied. However, it is well known that it is useful to regard the relative entropy as a distance between two probability measures. See e.g. Cover and Thomas (2006, p.18) in statistics. The relative entropy has been lately used as a cost of controlling probability measures in economics. See e.g. Hansen and Sargent (2007), Hansen et al. (2006), Sims (2003). Also, Delbaen et al. (2002) use it as a penalty in hedging contingent claims in mathematical finance.

by using θ_j and α_i for all i, j as follows:

$$\begin{aligned} H(\mathbb{Q}\,||\,\mathbb{P}) &= \mathbb{E}^{\mathbb{Q}}\left[\log\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right] = \mathbb{E}^{\mathbb{Q}}\left[\begin{array}{l} \sum_{j=1}^{n}\int_0^T \theta_j(s)\,\mathrm{d}B(s) - \frac{1}{2}\int_0^T(\theta_j(s))^2\,\mathrm{d}s + \\ \sum_{i=1}^{m}\int_0^T\left\{\widetilde{\lambda}_i(s)\log\left(\frac{\widetilde{\lambda}_i(s)}{\lambda_i}\right) + (\lambda_i - \widetilde{\lambda}_i(s))\right\}\mathrm{d}s \end{array}\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[\sum_{j=1}^{n}\int_0^T\frac{(\theta_j(s))^2}{2}\,\mathrm{d}s + \sum_{i=1}^{m}\int_0^T\left\{\widetilde{\lambda}_i(s)\log\left(\frac{\widetilde{\lambda}_i(s)}{\lambda_i}\right) + (\lambda_i - \widetilde{\lambda}_i(s))\right\}\mathrm{d}s\right] \quad (2.6) \end{aligned}$$

Recall that $\frac{\widetilde{\lambda}_i(s)}{\lambda_i} = \alpha_i(s) + 1$. If there are no jump terms, $H(\mathbb{Q}\,||\,\mathbb{P}) = \mathbb{E}^{\mathbb{Q}}\left[\sum_{j=1}^{n}\frac{1}{2}\int_0^T(\theta_j(s))^2\,\mathrm{d}s\right]$. The second term inside the expectation of Eq.(2.6) corresponds to the jump terms.

2.4 Consumption good

The players consume the consumption good at each instant over $[0, T]$ by sharing the produced consumption good. Let $S = \{S(t)\}_{0\le t\le T}$ and $C = \{C(t)\}_{0\le t\le T}$ denote the consumption processes of the firm and the investor, respectively. Let $\mathcal{H}$ denote the Hilbert space of all $\mathbf{R}$-valued adapted processes Y such that $\mathbb{E}^{\mathbb{P}}\left[\int_0^T Y(t)^2\,\mathrm{d}t + Y(T)^2\right] < \infty$ with the inner product $(Y|Z) := \mathbb{E}^{\mathbb{P}}\left[\int_0^T Y(t)Z(t)\,\mathrm{d}t + Y(T)Z(T)\right]$ for $Y, Z \in \mathcal{H}$. $\mathcal{H}_+$ denotes the set of all non-negative processes in $\mathcal{H}$. C, S are in $\mathcal{H}_+$. Also, the endowment process X characterized by Eq.(2.5) is in $\mathcal{H}_+$. Because of the non-storability attribute, the terminal stock levels of the consumptions are zeros, i.e., $C(T) = 0, S(T) = 0$.

The good is shared according to terms of a contract – call S a contract or a payment rule. S is the firm's consumption while C is the investor's one out of her allocation $X - S$. At time 0, the investor offers a contract $\{S(t)\}_{0\le t\le T}$ to the firm, and the firm then decides whether or not to accept it. We will specify below mathematical regularities for the sets of C, S.

2.5 Financial markets

Financial markets are accessible only to the investor, not to the firm. There are two types of financial securities in the markets: a riskless asset and d risky assets. The return process of the riskless asset with its price $\{P_0(t)\}_{0\le t\le T}$ is denoted by:

$$\frac{\mathrm{d}P_0(t)}{P_0(t)} = r(t)\,\mathrm{d}t$$

where $r(t)$ is the riskless rate and is determined endogenously in the markets. This model assumes that $r(t)$ can be negative and, in addition, $\int_0^T |r(t)|\,\mathrm{d}t < \infty\, a.s.$.

The excess returns of the risky assets, denoted by d-dimensional $\mathrm{d}R$, are characterized by:

$$\mathrm{d}R(t) = \mu^R\,\mathrm{d}t + \sum_{j=1}^{n} \sigma_j^R\,\mathrm{d}B_j(t) + \sum_{i=1}^{m} z_i^R\,\mathrm{d}M_i(t)$$

where the elements of $\mu^R \in \mathbf{R}^d$, $\sigma_j^R \in \mathbf{R}^d$ and $z_i^R \in \mathbf{R}^d\,\forall\, i,j$ are constants. In particular, the elements of $\sigma_j^R > 0\,\forall\, j$, and the elements of $z_i^R > -1\,\forall\, i$. We may also write a $(d\times n)$-matrix $\sigma^R := (\sigma_1^R,\cdots,\sigma_n^R)$ and a $(d\times m)$-matrix $z^R := (z_1^R,\cdots,z_m^R)$. Note that, for simplicity, we are assuming that the financial markets are under the same probability measure as the production.[16] By assuming that the representative firm controls the whole probability space, we examine the effect of real macro shocks on asset prices.

The investor has initial funds $W(0) = w_0 = 0$ at time 0 and makes a financial portfolio among the financial securities dynamically over $[0,T]$. This model presumes that the representative investor can take non-zero positions of the financial securities in off-equilibrium, but takes zero positions in equilibrium. To make the price process well-defined, we assume that the wealth process can be negative in the off-equilibrium, as in Cox et al. (1985).[17] The portfolio ratio on the risky assets is denoted by an $\mathbf{R}^d$-valued predictable process β such that:

Definition 2.1 *Define the set of β as $\mathcal{B}$ such that $\mathcal{B}$ is the set of $\mathbb{F}$-predictable processes β satisfying $\int_0^T |\beta(t)|^2\,\mathrm{d}t < +\infty\, a.s.$ and $\beta(t)^\top z_i^R \geq -1$, $\forall\, i$.*

The residual is invested in the riskless asset. Thus, the wealth process W is characterized by

$$\mathrm{d}W(t) = W(t_-)r(t)\,\mathrm{d}t + W(t_-)\beta(t)^\top\,\mathrm{d}R(t) + \Big(X(t) - C(t) - S(t)\Big)\,\mathrm{d}t, \quad W(0) = w_0. \tag{2.7}$$

The wealth is storable while the consumption good is not. The last term $\Big(X(t) - C(t) - S(t)\Big)$ on the right-hand side of Eq.(2.7) is the *current* value of the saving of the consumption good in the *future* spot market. It can be negative: the investor can borrow from the markets. In equilibrium, however, all the consumption good is used up and the saving is zero in the financial markets.

[16]This may look restrictive. We might assume, instead, that the firm controls only some part of the probability space.

[17]In contrast to Cox et al. (1985), there exist negative proportional jumps in this model. When a jump occurs for negative wealth, the wealth-level improves. This seems inconceivable. However, such negative wealth never takes places in equilibrium.

Since μ^R, σ^R, z^R are constants and $\int_0^T |\beta(t)|^2 \,\mathrm{d}t < +\infty \; a.s.$ because of Definition 2.1,

$$\int_0^T |\beta(t)^\top \mu^R| \,\mathrm{d}t < +\infty, \quad \int_0^T |\beta(t)^\top \sigma^R|^2 \,\mathrm{d}t < +\infty, \text{ and } \int_0^T |\beta(t)^\top z_i^R| \,\mathrm{d}t < +\infty \,\forall i, \; a.s..$$

Also,

$$\begin{aligned} \beta(t)^\top \mathrm{d}R(t) &= \sum_{k=1}^d \beta_k(t)\Big(\mu_k^R \,\mathrm{d}t + \sum_{j=1}^n \sigma_{k,j}^R \,\mathrm{d}B_j(t) + \sum_{i=1}^m z_{k,i}^R \,\mathrm{d}M_i(t)\Big) \\ &= \sum_{k=1}^d \beta_k(t)\mu_k^R \,\mathrm{d}t + \sum_{j=1}^n \Big\{ \sum_{k=1}^d \beta_k(t)\sigma_{k,j}^R \Big\} \mathrm{d}B_j(t) + \sum_{i=1}^m \Big\{ \sum_{k=1}^d \beta_k(t) z_{k,i}^R \Big\} \mathrm{d}M_i(t). \end{aligned}$$

The condition $\beta(t)^\top z_i^R \geq -1 \,\forall\, i$ ensures that the wealth does not turn into opposite signs by jumps.

2.6 Utility

The two players rank their own consumption/wealth processes via time-separable utility of consumption, characterized by a common constant instantaneous discount factor $\delta > 0$ and different instantaneous utility functions of consumption/wealth. Let $f_k : \mathbf{R}_{++} \to \mathbf{R} \cup \{-\infty\}$ for $k \in \{1, 2\}$ denote player k's instantaneous utility function of his or her own consumption. In particular, we set $f_1(x) = a \log x$ for $x > 0$ and $f_2(x) = \frac{x^{1-\gamma}}{1-\gamma}$ for $x > 0$ where a, γ are constants and $0 < \gamma < 1$. The investor's power utility with $0 < \gamma < 1$ means that she is less risk-averse than the firm. Obviously, for $k = 1, 2$, the utility function f_k is non-decreasing and concave, and is continuously differentiable on its effective domain denoted by $\mathrm{dom}\, f_k := \{x \in \mathbf{R} \,|\, f_k(x) > -\infty\}$. Also, the investor enjoys a linear utility of the terminal wealth $W(T)$ while the firm does not receive any utility of wealth.[18]

Now, the firm's performance criterion is written as follows. For the controlled probability measure $\mathbb{Q}$ and for the consumption $\{S(t)\}_{0 \leq t \leq T}$, let $U_1(\mathbb{Q};\, S)$ denote the firm's expected discounted utility, net of the effort cost:

$$U_1(\mathbb{Q};\, S) = \mathbb{E}^{\mathbb{Q}}[\int_0^T e^{-\delta u} \log S(u) \,\mathrm{d}u] - H(\mathbb{Q} \,||\, \mathbb{P}). \tag{2.8}$$

He is exogenously given a reservation utility, denoted by a constant $\rho \in \mathbf{R}$, at time 0. If the investor offers to the firm any lower utility than the reservation utility ρ ex ante, the firm would not take

[18]If the financial markets are accessible to the firm, the moral hazard problem would influence the firm's utility via the financial markets as well. With such accessibility, the firm might make higher efforts than otherwise, because, by doing so, he would receive higher financial returns.

the offer. To make the contractual relationship viable, we assume

Assumption 2.1 $\rho < \mathbb{E}^{\mathbb{P}}\left[\int_0^T e^{-\delta u} a \log X(u)\,\mathrm{d}u\right]$.

As we will discuss in detail below, with this assumption, there is some room for making the firm participate in the contact either under moral hazard or no moral hazard.

Next, the investor's performance criterion is written as follows. For $\mathbb{Q}$ and for the consumption $\{C(t)\}_{0\le t\le T}$ and the terminal wealth $W(T)$, let $U_2(C, W(T); \mathbb{Q})$ denote the investor's expected discounted utility. I.e.,

$$
\begin{aligned}
U_2(C, W(T); \mathbb{Q}) &= \mathbb{E}^{\mathbb{Q}}\left[\int_0^T e^{-\delta u}\frac{C(u)^{1-\gamma}}{1-\gamma}\,\mathrm{d}u + e^{-\delta T} W(T)\right] \\
&= \mathbb{E}^{\mathbb{P}}\left[\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\left\{\int_0^T e^{-\delta u}\frac{C(u)^{1-\gamma}}{1-\gamma}\,\mathrm{d}u + e^{-\delta T} W(T)\right\}\right].
\end{aligned}
$$

2.7 Sets of controls

To ensure that the players' performance criteria are well-defined, we put mathematical regularities for the control triples $(S, C, \beta) \in \mathcal{H}_+ \times \mathcal{H}_+ \times \mathcal{B}$:

Definition 2.2 *Define $\mathcal{A}$ as the set of the control triples $(S, C, \beta) \in \mathcal{H}_+ \times \mathcal{H}_+ \times \mathcal{B}$ such that*

(i) $0 < S(t) \le X(t)\ \forall\,[0,T)$ *a.s. and* $S(T) = 0$,

(ii) *S is of the Markovian form $S(t) := \tilde{s}(t, X(t))$ for some deterministic function $\tilde{s}$. In particular, we restrict the contract form as follows: for some constant $s \in (0,1) \subset \mathbf{R}_{++}$, $S(t) = sX(t)\,\forall\, t \in [0,T)$ a.s.. The contract is linear (i.e., proportional to the production) at a constant (i.e, time-independent) rate of change — call it a stationary linear payment rule. Note that it does not depend on W.*

(iii) $\mathbb{P}[\int_0^T e^{-\delta u}\log S(u)\,\mathrm{d}u > -\infty] > 0$,

(iv) $0 < C(t) \le X(t)\ \forall\,[0,T)$ *a.s. and* $C(T) = 0$,

(v) *C is of the Markovian form $C(t) := \tilde{c}(t, X(t), W(t))$ for some deterministic function $\tilde{c}$,*

(vi) *$\beta \in \mathcal{B}$ is of the Markovian form $\beta(t) := \tilde{b}(t, X(t), W(t))$ for some deterministic function $\tilde{b}$,*

(vii) *The wealth process W, which is generated via Eq.(2.7), satisfies $\mathbb{E}^{\mathbb{P}}[(W(T))^2] < \infty$.*

Because of the assumption of the Markovian controls, we avoid the measurability problem of the control functions in dynamic programing (see e.g. Pham (2009, p.42)). With regard to Definition 2.2 (ii), we can conjecture that the linearity assumption is not restrictive in optimum, because the entire system of equations is linear in this model.[19] On the other hand, however, the stationarity assumption is restrictive in the finite-horizon model. To examine the effect of non-stationarity, we will later perturb the results of equilibrium asset pricing under the stationary linear contract, by using the Taylor series expansion.

We check the integrability of U_1 and U_2 here. The process X, characterized by Eq.(2.5), is integrable. So is S owing to Definition 2.2 (i). By the concavity of f_1, $U_1 < +\infty$. On the other hand, Definition 2.2 (iii) does not always ensure $U_1 > -\infty$. Still, since the firm reserves $\rho \in \mathbf{R}$ as an outside option, U_1 is well-defined in equilibrium. With regard to U_2, because of the integrability of X and Definition 2.2 (iv), $U_2 < +\infty$. In addition, since $f_2 \geq 0$, $U_2 \geq 0$. Hence, U_2 is well-defined.

3 Optimization

3.1 Firm's optimization

Let $\mathcal{S}$ denote the set of S that satisfies Definition 2.2 (i), (ii), (iii). For a contact $S \in \mathcal{S}$, we formulate the firm's optimal expected utility under the controlled probability measure $\mathbb{Q}$, denoted by V_1, as:

$$V_1 := \sup_{\substack{\mathbb{Q} \ll \mathbb{P} \\ H(\mathbb{Q} || \mathbb{P}) < \infty}} U_1 \left(\mathbb{Q}; S \right). \tag{3.1}$$

We obtain the following lemma:

Lemma 3.1 *For $S \in \mathcal{S}$,*

$$V_1 = \log \mathbb{E}^{\mathbb{P}} \left[e^{\int_0^T e^{-\delta u} a \log S(u) \, \mathrm{d}u} \right]. \tag{3.2}$$

The maximizer, denoted by $\mathbb{Q}^*$, is then characterized by an L_2-process:

$$\frac{\mathrm{d}\mathbb{Q}^*}{\mathrm{d}\mathbb{P}} = \frac{e^{\int_0^T e^{-\delta u} a \log S(u) \, \mathrm{d}u}}{\mathbb{E}^{\mathbb{P}} \left[e^{\int_0^T e^{-\delta u} a \log S(u) \, \mathrm{d}u} \right]} = e^{-V_1} e^{\int_0^T e^{-\delta u} a \log S(u) \, \mathrm{d}u}. \tag{3.3}$$

[19] As we will discuss below, if the firm's instantaneous utility function is not of a logarithm type, then the linearity of the entire system of equations is not ensured.

This result and its variants are known in the fields of operations research and mathematical finance: for a literature review, see e.g. the first remark in Section 1 of Delbaen et al. (2002). For the sake of completeness, we present a proof.

Proof: See appendix.

The investor cannot directly observe the true probability measure $\mathbb{Q}$, but can verify the optimal $\mathbb{Q}^*$ by designing the contract to satisfy Eq.(3.3). Thus, the investor can implement the optimal $\mathbb{Q}^*$ by controlling S through Eq.(3.3). To ensure that the firm participates in the contract, the investor provides him with no lower utility than his reservation utility, i.e.,

$$V_1 \geq \rho. \tag{3.4}$$

We call it the participation constraint. From Eq.(3.3) and Eq.(3.4),

$$\frac{\mathrm{d}\mathbb{Q}^*}{\mathrm{d}\mathbb{P}} = e^{-V_1} e^{\int_0^T e^{-\delta u} a \log S(u)\,\mathrm{d}u} \leq e^{-\rho} e^{\int_0^T e^{-\delta u} a \log S(u)\,\mathrm{d}u} \tag{3.5}$$

with equality if the participation constraint is binding (i.e., $V_1 = \rho$). Eq.(3.5) works as an incentive constraint in the investor's optimization, as shown in the next subsection.

3.2 Investor's optimization

We next look at the investor's optimization. Taking as given the firm's optimization characterized by Lemma 3.1, the investor optimizes his utility while controlling $\mathbb{Q}$ indirectly by giving the firm the incentive Eq.(3.5). In addition, the investor's optimization is subject to the firm's participation constraint Eq.(3.4), i.e., the investor gives the firm no lower utility than his reservation utility ρ. Let $\{V_2(t)\}_{0\leq t\leq T}$ denote the process of the investor's optimal expected discounted utility. We formulate the investor's optimization problem as follows:

$$V_2(0) \quad = \sup_{(S,C,\beta)\in\mathcal{A}} U_2\left(C, W(T)\,;\,\mathbb{Q}^*\right) \tag{3.6}$$

$$
\begin{aligned}
\text{s.t. } \mathrm{d}W(t) &= W(t_-)r(t)\,\mathrm{d}t + W(t_-)\beta(t)^\top \mathrm{d}R(t) + \Big(X(t) - C(t) - S(t)\Big)\,\mathrm{d}t,\ W(0) = w_0, \\
\mathrm{d}R(t) &= \mu^R\,\mathrm{d}t + \sum_{j=1}^{n} \sigma_j^R\,\mathrm{d}B_j(t) + \sum_{i=1}^{m} z_i^R\,\mathrm{d}M_i(t), \\
\mathrm{d}X(t) &= X(t_-)\Big(\mu^G\,\mathrm{d}t + \sum_{j=1}^{n} \sigma_j^G\,\mathrm{d}B_j(t) + \sum_{i=1}^{m} z_i^G\,\mathrm{d}M_i(t)\Big),\ X(0) = x_0, \\
V_1 &= \log \mathbb{E}^{\mathbb{P}}\Big[e^{\int_0^T e^{-\delta u} a \log S(u)\,\mathrm{d}u}\Big] \geq \rho, && \text{(from Eq.(3.4))} \\
\frac{\mathrm{d}\mathbb{Q}^*}{\mathrm{d}\mathbb{P}} &= e^{-V_1} e^{\int_0^T e^{-\delta u} a \log S(u)\,\mathrm{d}u} && \text{(from Eq.(3.5))}
\end{aligned}
$$

where $U_2\left(C, W(T)\,;\,\mathbb{Q}^*\right) = \mathbb{E}^{\mathbb{Q}^*}\left[\int_0^T e^{-\delta u}\frac{C(u)^{1-\gamma}}{1-\gamma}\,\mathrm{d}u + e^{-\delta T}W(T)\right]$. S is designed ex ante while the pair $(C,\, \beta)$ is controlled ex post. Owing to Eq.(3.5), the investor can take her expectation under $\mathbb{Q}^*$.

Definition 3.1 *A control triple $(S, C, \beta) \in \mathcal{A}$ is said to be optimal for Eq.*(3.6) *if $(S, C, \beta) \in \mathcal{A}$ is the maximizer of Eq.*(3.6)*, if any.*

From Eq.(3.6), we define the Lagrangian as follows:

$$
\begin{aligned}
& \sup_{(S,C,\beta)\in\mathcal{A}} \mathbb{E}^{\mathbb{Q}^*}\left[\int_0^T e^{-\delta u}\frac{C(u)^{1-\gamma}}{1-\gamma}\,\mathrm{d}u + e^{-\delta T}W(T) + \chi\right] \qquad (3.7) \\
= & \sup_{(S,C,\beta)\in\mathcal{A}} e^{-V_1}\mathbb{E}^{\mathbb{P}}\left[e^{\int_0^T e^{-\delta u} a \log S(u)\,\mathrm{d}u}\left(\int_0^T e^{-\delta u}\frac{C(u)^{1-\gamma}}{1-\gamma}\,\mathrm{d}u + e^{-\delta T}W(T) + \chi\right)\right] \\
\text{s.t.} \quad & \mathrm{d}W(t) = W(t_-)r(t)\,\mathrm{d}t + W(t_-)\beta(t)^\top \mathrm{d}R(t) + \Big(X(t) - C(t) - S(t)\Big)\,\mathrm{d}t, \quad W(0) = w_0, \\
& \mathrm{d}R(t) = \mu^R\,\mathrm{d}t + \sum_{j=1}^{n} \sigma_j^R\,\mathrm{d}B_j(t) + \sum_{i=1}^{m} z_i^R\,\mathrm{d}M_i(t), \\
& \mathrm{d}X(t) = X(t_-)\Big(\mu^G\,\mathrm{d}t + \sum_{j=1}^{n} \sigma_j^G\,\mathrm{d}B_j(t) + \sum_{i=1}^{m} z_i^G\,\mathrm{d}M_i(t)\Big), \quad X(0) = x_0
\end{aligned}
$$

where χ denotes the Lagrangian multiplier associated with the participation constraint Eq.(3.4).

4 Market equilibrium

4.1 Definition

We define market equilibrium:

Definition 4.1 *A control triple $(S, C, \beta) \in \mathcal{A}$ is said to be in market equilibrium if the following conditions are satisfied:*

(i) *The control triple $(S, C, \beta) \in \mathcal{A}$ is optimal for Eq.*(3.6)*,*

(ii) *The market of the good is cleared, i.e., for all t, $C(t) = X(t) - S(t)$,*

(iii) *The markets of the financial assets are cleared, i.e., for all t, all the elements of $\beta(t) = 0$.*

4.2 Characterization

If the payment rule S takes more general forms than the stationary linear one, then it would be hard to obtain an explicit solution either analytically or numerically to the investor's optimization problem subject to the two constraints on states: Eq.(3.4) and Eq.(3.5).[20] However, we can solve the problem because of the stationary linear payment rule. The Lagrangian is rewritten as:

$$\begin{aligned}
&\sup_{(S,C,\beta)\in\mathcal{A}} \mathbb{E}^{\mathbb{P}}\left[\frac{e^{\int_0^T e^{-\delta u} a \log X(u)\,\mathrm{d}u}}{\mathbb{E}^{\mathbb{P}}\left[e^{\int_0^T e^{-\delta u} a \log X(u)\,\mathrm{d}u}\right]}\left(\int_0^T e^{-\delta u}\frac{C(u)^{1-\gamma}}{1-\gamma}\,\mathrm{d}u + e^{-\delta T} W(T) + \chi\right)\right] \\
= &\sup_{(S,C,\beta)\in\mathcal{A}} \left(\begin{array}{l} \mathbb{E}^{\mathbb{P}}\left[e^{\int_0^T e^{-\delta u} a \log X(u)\,\mathrm{d}u}\right]^{-1} \cdot \\ \mathbb{E}^{\mathbb{P}}\left[\begin{array}{l} \int_0^T e^{-\delta u}\mathbb{E}_u^{\mathbb{P}}[e^{\int_0^T e^{-\delta u} a \log X(u)\,\mathrm{d}u}]\frac{C(u)^{1-\gamma}}{1-\gamma}\,\mathrm{d}u \\ +e^{\int_0^T e^{-\delta u} a \log X(u)\,\mathrm{d}u}\left(e^{-\delta T} W(T) + \chi\right) \end{array}\right] \end{array}\right) \qquad (4.1) \\
\text{s.t. } \mathrm{d}W(t) &= W(t_-) r(t)\,\mathrm{d}t + W(t_-)\beta(t)^\top \mathrm{d}R(t) + \Big(X(t) - C(t) - S(t)\Big)\,\mathrm{d}t, \quad W(0) = w_0, \\
\mathrm{d}R(t) &= \mu^R\,\mathrm{d}t + \sum_{j=1}^n \sigma_j^R\,\mathrm{d}B_j(t) + \sum_{i=1}^m z_i^R\,\mathrm{d}M_i(t), \\
\mathrm{d}X(t) &= X(t_-)\Big(\mu^G\,\mathrm{d}t + \sum_{j=1}^n \sigma_j^G\,\mathrm{d}B_j(t) + \sum_{i=1}^m z_i^G\,\mathrm{d}M_i(t)\Big), \quad X(0) = x_0.
\end{aligned}$$

Define

$$Y(t) := \mathbb{E}_t^{\mathbb{P}}\left[e^{a\int_0^T e^{-\delta u}\log X(u)\,\mathrm{d}u}\right]. \qquad (4.2)$$

As seen in Eq.(4.1), $Y(t)$ represents a time-t distortion of the probability measure caused by the endowment process X all over $[0, T]$. We obtain the following lemma:

[20] With regard to such mathematical difficulty, see e.g. Yong and Zhou (1999, p.155).

Lemma 4.1

$$Y(t) = \exp\left(\begin{array}{l} a\left(\begin{array}{l} \int_0^t e^{-\delta u} \log X(u)\,\mathrm{d}u + \frac{e^{-\delta t}-e^{-\delta T}}{\delta} \log X(t) \\ +e^{-\delta t}\left(\mu^G - \frac{1}{2}\sum_{j=1}^n (\sigma_j^G)^2 - \sum_{i=1}^m \lambda_i\right) \frac{1-e^{-\delta(T-t)}(1+\delta(T-t))}{\delta^2} \end{array} \right) \\ +\sum_{j=1}^n \frac{(ae^{-\delta t}\sigma_j^G)^2}{2} \frac{1}{\delta^2} \int_0^{T-t} (e^{-\delta u} - e^{-\delta(T-u)})^2\,\mathrm{d}u \\ +\sum_{i=1}^m \lambda_i (T-t)(-1 + e^{-\frac{e^{-\delta(T-t)}}{\delta} \log(1+z_i^G)} \frac{1}{T-t} \int_0^{T-t} e^{\frac{e^{-\delta u}}{\delta} \log(1+z_i^G)}\,\mathrm{d}u) \end{array} \right).$$

Proof : See appendix.

We next characterize the stochastic differential of $Y(t)$. Noting that $Y(t)$ is a martingale,

Lemma 4.2 *The martingale $Y(t)$ satisfies*

$$\mathrm{d}Y(t) = Y(t_-)\left\{ a\frac{e^{-\delta t} - e^{-\delta T}}{\delta} \sum_{j=1}^n \sigma_j^G\,\mathrm{d}B_j(t) + \sum_{i=1}^m \left((1+z_i^G)^{a\frac{e^{-\delta t}-e^{-\delta T}}{\delta}} - 1 \right) \mathrm{d}M_i(t) \right\}.$$

Proof: See appendix.

Note that it is because of the firm's log utility function that we have obtained such explicit characterization of Y. If the firm's utility takes on types of forms other than the logarithm type, including the power type, then we would not obtain the linearity of the stochastic differential of Y.

As s is not dynamic, we solve Problem (4.1) by using Y as well as X, W as state variables for some appropriate s given. We apply a verification theorem for the optimal control problem with the classical Hamilton-Jacobi-Bellman (HJB) equation (see e.g. Øksendal and Sulem (2007, Theorem 3.1, p.46), Pham (2009, Theorem 3.5.2, p.47)). For the given s and for a twice continuously differentiable function $h(t, X(t), Y(t), W(t)) \in C^{1,2,2,2}$, define the generator of $(t, X(t), Y(t), W(t))$ as:

$$\begin{aligned} & \mathcal{L}^{C,\beta} h(t, X(t), Y(t), W(t)) \qquad (4.3) \\ := \; & h_t + \mu^G X(t) h_x + \Big(rW(t) + \beta(t)^\top \mu^R W(t) + \Big((1-s)X(t) - C(t)\Big)\Big) h_w \\ + \; & \frac{1}{2}\sum_{j=1}^n \left\{ (\sigma_j^G)^2 X(t)^2 h_{xx} + (\beta(t)^\top \sigma_j^R)^2 W(t)^2 h_{ww} + (a\frac{e^{-\delta t} - e^{-\delta T}}{\delta}\sigma_j^G)^2 Y(t)^2 h_{yy} \right\} \\ + \; & \sum_{j=1}^n \left\{ \begin{array}{l} (\beta(t)^\top \sigma_j^R \sigma_j^G) X(t) W(t) h_{xw} + (a\frac{e^{-\delta t}-e^{-\delta T}}{\delta}(\sigma_j^G)^2) X(t) Y(t) h_{xy} \\ +(a\frac{e^{-\delta t}-e^{-\delta T}}{\delta}\beta(t)^\top \sigma_j^R \sigma_j^G) Y(t) W(t) h_{yw} \end{array} \right\} \\ + \; & \sum_{i=1}^m \lambda_i \left\{ \begin{array}{l} h(t, (1+z_i^G)X(t), (1+z_i^G)^{a\frac{e^{-\delta t}-e^{-\delta T}}{\delta}} Y(t), (1+\beta(t)^\top z_i^R) W(t)) \\ -h(t, X(t), Y(t), W(t)) \end{array} \right\} \end{aligned}$$

where $h_t := \frac{\partial h}{\partial t}$, $h_x := \frac{\partial h}{\partial X(t)}$, $h_y := \frac{\partial h}{\partial Y(t)}$, $h_w := \frac{\partial h}{\partial W(t)}$, $h_{xx} := \frac{\partial^2 h}{\partial X(t)^2}$, $h_{yy} := \frac{\partial^2 h}{\partial Y(t)^2}$, $h_{ww} := \frac{\partial^2 h}{\partial W(t)^2}$, $h_{xy} := \frac{\partial^2 h}{\partial X(t)\partial Y(t)}$, $h_{yw} := \frac{\partial^2 h}{\partial Y(t)\partial W(t)}$ and $h_{xw} := \frac{\partial^2 h}{\partial X(t)\partial W(t)}$ that are continuous.

For notational convenience, using the result of Lemma 4.1, define a constant K as

$$K := \frac{1}{Y(0)} = \frac{1}{\mathbb{E}^{\mathbb{P}}\left[e^{\int_0^T e^{-\delta u} a \log X(u)\,\mathrm{d}u}\right]}.$$

By the verification theorem, if, for the given s and for $J(t, X(t), Y(t), W(t)) \in C^{1,2,2,2}$,

$$0 = \sup_{C(t),\beta(t)} \left(K\,Y(t)\frac{C(t)^{1-\gamma}}{1-\gamma} + \mathcal{L}^{C,\beta} J(t, X(t), Y(t), W(t)) \right) \tag{4.4}$$

holds under some regularities, then $V_2(t) = J(t, X(t), Y(t), W(t))$.

From Eq.(4.4), the lower s is better for the investor so long as the participation constraint is satisfied. In optimum, the participation constraint binds. Let $s^* \in (0, 1)$ denote the optimal sharing ratio. Because, from Assumption 2.1, $\rho < \mathbb{E}^{\mathbb{P}}\left[\int_0^T e^{-\delta u} a \log X(u)\,\mathrm{d}u\right] \leq \log \mathbb{E}^{\mathbb{P}}\left[e^{\int_0^T e^{-\delta u} a \log X(u)\,\mathrm{d}u}\right]$ by Jensen's inequality, there exists s^* such that

$$\begin{aligned} \rho &= \frac{(1-e^{-\delta T})a \log s^*}{\delta} + \log \mathbb{E}^{\mathbb{P}}\left[e^{\int_0^T e^{-\delta u} a \log X(u)\,\mathrm{d}u}\right]. \\ \text{I.e.,}\quad s^* &= \exp\left\{ \frac{\delta}{a(1-e^{-\delta T})}\left(\rho - \log \mathbb{E}^{\mathbb{P}}\left[e^{\int_0^T e^{-\delta u} a \log X(u)\,\mathrm{d}u}\right]\right)\right\}. \end{aligned} \tag{4.5}$$

The term $\log \mathbb{E}^{\mathbb{P}}\left[e^{\int_0^T e^{-\delta u} a \log X(u)\,\mathrm{d}u}\right]$ stands for the firm's optimal utility in the case that he would receive the whole production over time. If $\log \mathbb{E}^{\mathbb{P}}\left[e^{\int_0^T e^{-\delta u} a \log X(u)\,\mathrm{d}u}\right] < \rho$, the firm would not obtain a level of the reservation utility under the contract even if he receives the whole production over time. From the result of Lemma 4.1, plugging $Y(0) = \mathbb{E}^{\mathbb{P}}\left[e^{\int_0^T e^{-\delta u} a \log X(u)\,\mathrm{d}u}\right]$ into Eq.(4.5),

$$s^* = \exp\left\{ \begin{array}{l} \frac{\delta}{a(1-e^{-\delta T})}\times \\ \left(\rho - \left(\begin{array}{l} a\left(\begin{array}{l} \frac{1-e^{-\delta T}}{\delta}\log x_0 \\ +\left(\mu^G - \frac{1}{2}\sum_{j=1}^n (\sigma_j^G)^2 - \sum_{i=1}^m \lambda_i\right)\frac{1-e^{-\delta T(1+\delta T)}}{\delta^2} \end{array}\right) \\ +\sum_{j=1}^n \frac{(a\sigma_j^G)^2}{2}\frac{1}{\delta^2}\int_0^T (e^{-\delta u} - e^{-\delta T})^2\,\mathrm{d}u \\ +\sum_{i=1}^m \lambda_i T(-1 + e^{-\frac{e^{-\delta T}}{\delta}\log(1+z_i^G)}\frac{1}{T}\int_0^T e^{\frac{e^{-\delta u}}{\delta}\log(1+z_i^G)}\,\mathrm{d}u) \end{array}\right)\right) \end{array}\right\}. \tag{4.6}$$

From the HJB equation (4.4), we obtain a necessary and sufficient condition for optimality:

$$[C(t)] \quad K\,Y(t)C(t)^{-\gamma} = J_w > 0, \tag{4.7}$$

$$[\beta(t)] \quad W(t)J_w(\mu^R)^\top + W(t)^2 J_{ww}\beta(t)^\top\sigma^R(\sigma^R)^\top + W(t)X(t)J_{xw}\sigma^G(\sigma^R)^\top + W(t)Y(t)J_{wy}\sigma^G(\sigma^R)^\top a\frac{e^{-\delta t}-e^{-\delta T}}{\delta} + W(t)J_w\lambda^\top(z^R)^\top = 0. \tag{4.8}$$

Using these equations, we characterize the equilibrium solution. With regard to β, dividing both sides of Eq.(4.8) by $W(t)$ and taking the limit $W(t)\to 0$ owing to the market clearing condition, the equilibrium excess return is characterized by

$$\mu^R + z^R\lambda = -\sigma^R(\sigma^G)^\top\frac{X(t)J_{xw} + Y(t)J_{wy}a\frac{e^{-\delta t}-e^{-\delta T}}{\delta}}{J_w}. \tag{4.9}$$

Next, we look at the consumption/wealth. The equilibrium consumption processes of the investor and the firm, denoted by C^* and S^* respectively, are written as: using s^* defined by Eq.(4.6),

$$C^*(t) = (1-s^*)X(t), \tag{4.10}$$

$$S^*(t) = s^*X(t). \tag{4.11}$$

Also, the equilibrium wealth, denoted by W^*, is: $W^*(t) = 0\,\forall\,t$ a.s..

We characterize the investor's value function in the market equilibrium. Since $W^*(t) = 0\,\forall\,t$ a.s., it is sufficient to consider that the value function is $J(t, X(t), Y(t), 0)$ in the market equilibrium. With regard to the generator Eq.(4.3) for $h = J$, the J_w-term in the market equilibrium is

$$\left(r\cdot 0 + \beta(t)^\top\mu^R\cdot 0 + \Big((1-s^*)X(t) - C^*(t)\Big)\right)J_w(t, X(t), Y(t), 0) = 0.$$

Let the value function be denoted by $J^*(t, X(t), Y(t)) := J(t, X(t), Y(t), 0)$. From Eq.(4.4), for $x = X(t)$, $y = Y(t)$,

$$\begin{aligned} 0 &= K(1-s^*)^{1-\gamma}y\frac{x^{1-\gamma}}{1-\gamma} + J_t^* + \mu^G x J_x^* \\ &+ \frac{1}{2}\sum_{j=1}^n\left\{(\sigma_j^G)^2x^2J_{xx}^* + (a\frac{e^{-\delta t}-e^{-\delta T}}{\delta}\sigma_j^G)^2y^2J_{yy}^*\right\} + \sum_{j=1}^n a\frac{e^{-\delta t}-e^{-\delta T}}{\delta}(\sigma_j^G)^2xyJ_{xy}^* \\ &+ \sum_{i=1}^m\lambda_i\left\{J^*(t, (1+z_i^G)x, (1+z_i^G)^{a\frac{e^{-\delta t}-e^{-\delta T}}{\delta}}y) - J^*(t, x, y)\right\}. \end{aligned}$$

We try $J^*(t,x,y) = py\frac{x^{1-\gamma}}{1-\gamma}$ for some deterministic function of time $p = p(t)$ with $p(T) = 0$. Thus,

$$p' + pL(t) = -K(1-s^*)^{1-\gamma}; \quad p(T) = 0$$

$$\text{where} \quad L(t) := (1-\gamma)\mu^G - (1-\gamma)\sigma^G(\sigma^G)^\top \left(\frac{\gamma}{2} - a\frac{e^{-\delta t} - e^{-\delta T}}{\delta}\right) + \sum_{i=1}^{m} \lambda_i \left\{(1+z_i^G)^{(1-\gamma)+a\frac{e^{-\delta t}-e^{-\delta T}}{\delta}} - 1\right\}.$$

Hence, for some constant C and for $L(t) \neq 0$,

$$p(t) = Ce^{-\int L(t)\,\mathrm{d}t} - \frac{K(1-s^*)^{1-\gamma}}{L(t)}$$

$$\begin{aligned}\text{where } \int L(t)\,\mathrm{d}t &= \left((1-\gamma)\mu^G - (1-\gamma)\sigma^G(\sigma^G)^\top \left(\frac{\gamma}{2} + \frac{a}{\delta}e^{-\delta T}\right) - \sum_{i=1}^{m}\lambda_i\right)t \\ &\quad -(1-\gamma)\sigma^G(\sigma^G)^\top \frac{a}{\delta^2}e^{-\delta t} \\ &\quad - \frac{1}{\left((2-\gamma) + a\frac{e^{-\delta t}-e^{-\delta T}}{\delta}\right)ae^{-\delta t}} \sum_{i=1}^{m} \lambda_i (1+z_i^G)^{(2-\gamma)+a\frac{e^{-\delta t}-e^{-\delta T}}{\delta}}\end{aligned}$$

From $p(T) = 0$,

$$p(t) = \frac{K(1-s^*)^{1-\gamma}}{L(t)} \left(\frac{L(t)\exp\left\{-\int L(t)\,\mathrm{d}t\right\}}{L(T)\exp\left\{-\left(\int L(t)\,\mathrm{d}t\right)_{t=T}\right\}} - 1\right)$$

$$\begin{aligned}\text{where} \quad L(T) &= (1-\gamma)\mu^G - \frac{\gamma(1-\gamma)}{2}\sigma^G(\sigma^G)^\top + \sum_{i=1}^{m}\lambda_i\left\{(1+z_i^G)^{1-\gamma} - 1\right\}, \\ \left(\int L(t)\,\mathrm{d}t\right)_{t=T} &= \left((1-\gamma)\mu^G - (1-\gamma)\sigma^G(\sigma^G)^\top\left(\frac{\gamma}{2} + \frac{a}{\delta}e^{-\delta T}\right) - \sum_{i=1}^{m}\lambda_i\right)T \\ &\quad -(1-\gamma)\sigma^G(\sigma^G)^\top\frac{a}{\delta^2}e^{-\delta T} - \frac{1}{(2-\gamma)ae^{-\delta T}}\sum_{i=1}^{m}\lambda_i(1+z_i^G)^{2-\gamma}.\end{aligned}$$

Thus, we obtain the value function $J^*(t,x,y)$ explicitly. From Eq.(4.9),

$$\mu^R + z^R\lambda = -\sigma^R(\sigma^G)^\top\left(-\gamma + a\frac{e^{-\delta t} - e^{-\delta T}}{\delta}\right).$$

5 Equilibrium asset prices

We characterize the behavior of asset prices in the market equilibrium. Since the assets are traded ex post (i.e., after the contract is made), we consider the equilibrium behavior of asset prices by taking as given the optimally designed contract S^* defined by Eq.(4.11). For notational convenience, define the process of the investor's consumption/terminal wealth process $\phi \in \mathcal{H}_+$ as

$$\phi(t) := \begin{cases} C(t) & \text{for } 0 \le t < T \\ W(T) & \text{for } t = T \end{cases}$$

Since S^* defined by Eq.(4.11) is independent of W and β, the consumption/wealth process ϕ is determined uniquely for each $(S^*, C, \beta) \in \mathcal{A}$ and is thus well-defined in $\mathcal{A}$ with S^* given. Let $\Phi(S^*)$ denote the set of (ϕ, β) that corresponds to $(S^*, C, \beta) \in \mathcal{A}$.

In the following, we will first define a state price, say Π. I.e., $\Pi(t;\omega)$ is the price of a security that agrees to pay one unit of the consumption/wealth ϕ on a particular time path ω and to pay zero on others. Next, we will find optimal controls $(\phi, \beta) \in \Phi(S^*)$ – call them ex-post optimal controls – and then obtain an equilibrium state price. Finally, we will obtain a riskless rate and the market prices of diffusive and jump risks in equilibrium.[21]

5.1 State prices

We give a formal definition of state prices in the financial markets. Define a process $\Pi \in \mathcal{H}_+$ for some n dimensional process η and m dimensional process ξ as:

$$\mathrm{d}\Pi(t) = \Pi(t_-)\Big(- r(t)\,\mathrm{d}t - \eta(t)^\top \,\mathrm{d}B(t) - \xi(t)^\top \,\mathrm{d}M(t)\Big), \quad \Pi(0) = 1 \tag{5.1}$$

where η stands for the market price of diffusive risk and ξ stands for the market price of jump risk. Define an $(m \times m)$-matrix I^ξ in which its diagonal element $x_{ii} = \xi_i$ for all $i = 1, \cdots, m$ and the other elements $x_{ij} = 0$ for $i, j = 1, \cdots, m$ and $i \neq j$.

Definition 5.1 Π *is said to be a state price at a pair* $(\phi, \beta) \in \Phi(S^*)$ *if, for the pair* $(\phi, \beta) \in \Phi(S^*)$ *and for any* $h \in \mathcal{H}$ *such that* $(\phi + h, \beta') \in \Phi(S^*)$, $(\Pi|h) \le 0$.[22]

[21] This method is similar to the one in Skiadas (2007). However, there exists two main departures from that. First, we explore an SDE method, rather than a backward SDE method. Second, our method deals with jumps as well.

[22] We are defining Π over Ω, not at a particular ω. In this sense, it may be strictly called a state

We put an assumption on the market opportunities as follows.

Assumption 5.1 *For the market prices of diffusive and jump risks η, ξ, the law of one price holds in the markets. That is,*

$$\mu^R = \sigma^R \eta + z^R I^{\xi} \lambda.$$

From Eq.(2.7),

$$\begin{aligned} \mathrm{d}W(t) &= -\Big(C(t) + S^*(t) - X(t) - W(t_-)r(t) - W(t_-)\beta(t)^\top \mu^R\Big)\,\mathrm{d}t \\ &\quad + W(t_-)\beta(t)^\top \left(\sigma^R \,\mathrm{d}B(t) + z^R \,\mathrm{d}M(t)\right), \quad W(0) = w_0. \end{aligned}$$

Thus, by Itô's formula,

$$\begin{aligned} \mathrm{d}(\Pi(t)W(t)) &= W(t_-)\,\mathrm{d}\Pi(t) + \Pi(t_-)\,\mathrm{d}W(t) + \mathrm{d}\Pi(t)\,\mathrm{d}W(t) \\ &= \left\{ \begin{array}{l} -\Pi(t_-)\Big(C(t) + S^*(t) - X(t)\Big) \\ +\Pi(t_-)W(t_-)\beta(t)^\top \mu^R - \Pi(t_-)W(t_-)\beta(t)^\top \sigma^R \eta(t) \end{array} \right\} \mathrm{d}t \\ &\quad -\Pi(t_-)W(t_-)\beta(t)^\top z^R I^{\xi}(t)\,\mathrm{d}N(t) + \cdots \mathrm{d}B(t) + \cdots \mathrm{d}M(t). \end{aligned}$$

Assumption 5.2 *For a pair $(\phi, \beta) \in \Phi(S^*)$,*

$$\mathbb{E}^{\mathbb{P}}[\sup_t \Pi(t)W(t)] < \infty.$$

Lemma 5.1 *Fix S^* defined by Eq.*(4.11)*. Under Assumption 5.1,* Π *is a state price at the pair* $(\phi, \beta) \in \Phi(S^*)$ *satisfying Assumption 5.2.*

Proof: See appendix.

5.2 Optimal ex-post consumption/wealth and equilibrium state price

Under the contract S^* defined by Eq.(4.11), the investor controls her consumption/wealth ex post optimally while trading the assets. Let $\widehat{\mathbb{Q}}^*$ denote the optimal $\mathbb{Q}$ under $S = S^*$ when Eq.(3.4)

---

price density.

binds. From Eq.(3.5),

$$\frac{\mathrm{d}\widehat{\mathbb{Q}}^*}{\mathrm{d}\mathbb{P}} = e^{-\rho} e^{\int_0^T e^{-\delta u} a \log S^*(u)\,\mathrm{d}u} = e^{-\rho} e^{\frac{1-e^{-\delta T}}{\delta} a \log s^*} e^{\int_0^T e^{-\delta u} a \log X(u)\,\mathrm{d}u}.$$

From Eq.(2.9), let $\{\widehat{U}_2(t)\}_{0\le t\le T}$ denote the investor's utility process when Eq.(3.4) binds:

$$\widehat{U}_2(t) := e^{\delta t}\mathbb{E}^{\mathbb{P}}_t\left[\int_t^T e^{-\delta u}\mathbb{E}^{\mathbb{P}}_u[\frac{\mathrm{d}\widehat{\mathbb{Q}}^*}{\mathrm{d}\mathbb{P}}]\frac{C(u)^{1-\gamma}}{1-\gamma}\,\mathrm{d}u + e^{-\delta T}\frac{\mathrm{d}\widehat{\mathbb{Q}}^*}{\mathrm{d}\mathbb{P}}W(T)\right]$$

At $t = 0$, $\widehat{U}_2(0) = U_2(C, W(T);\, \widehat{\mathbb{Q}}^*)$ from Eq.(2.9). Also, at $t = T$, $\widehat{U}_2(T) = W(T)$. Since $e^{-\delta t}\widehat{U}_2(t)+\int_0^t e^{-\delta u}\mathbb{E}^{\mathbb{P}}_u[\frac{\mathrm{d}\widehat{\mathbb{Q}}^*}{\mathrm{d}\mathbb{P}}]\frac{C(u)^{1-\gamma}}{1-\gamma}\,\mathrm{d}u = \mathbb{E}^{\mathbb{P}}_t\left[\int_0^T e^{-\delta u}\mathbb{E}^{\mathbb{P}}_u[\frac{\mathrm{d}\widehat{\mathbb{Q}}^*}{\mathrm{d}\mathbb{P}}]\frac{C(u)^{1-\gamma}}{1-\gamma}\,\mathrm{d}u + e^{-\delta T}\frac{\mathrm{d}\widehat{\mathbb{Q}}^*}{\mathrm{d}\mathbb{P}}W(T)\right]$, it is a martingale. By the Martingale Representation Theorem again, there exist $\mathbb{F}$-predicable processes Θ_j and $\Xi_i \ge -1$, where $\int_0^T(\Theta_j(t))^2\,\mathrm{d}t < \infty$ and $\int_0^T \Xi_i(t)\,\mathrm{d}t < \infty$, for all i, j such that

$$\begin{aligned} e^{-\delta t}\widehat{U}_2(t) + \int_0^t e^{-\delta u}\mathbb{E}^{\mathbb{P}}_u[\frac{\mathrm{d}\widehat{\mathbb{Q}}^*}{\mathrm{d}\mathbb{P}}]\frac{C(u)^{1-\gamma}}{1-\gamma}\,\mathrm{d}u \;&=\; \mathbb{E}^{\mathbb{P}}\left[\int_0^T e^{-\delta u}\mathbb{E}^{\mathbb{P}}_u[\frac{\mathrm{d}\widehat{\mathbb{Q}}^*}{\mathrm{d}\mathbb{P}}]\frac{C(u)^{1-\gamma}}{1-\gamma}\,\mathrm{d}u + e^{-\delta T}\frac{\mathrm{d}\widehat{\mathbb{Q}}^*}{\mathrm{d}\mathbb{P}}W(T)\right] \\ &\quad + \int_0^T\sum_{j=1}^n \Theta_j(u)\,\mathrm{d}B_j(u) + \int_0^T\sum_{i=1}^m \Xi_i(u)\,\mathrm{d}M_i(u). \end{aligned}$$

I.e., there exist $\mathbb{F}$-predicable processes Σ and Γ such that

$$\begin{aligned} \mathrm{d}\widehat{U}_2(t) &= -\widehat{F}(t,\phi(t),\widehat{U}_2(t);\widehat{\mathbb{Q}}^*)\,\mathrm{d}t + \Sigma(t)\,\mathrm{d}B(t) + \Gamma(t)\,\mathrm{d}M(t), \\ \widehat{U}_2(T) &= \widehat{F}(T,\phi(T),\widehat{U}_2(T);\,\widehat{\mathbb{Q}}^*) \end{aligned}$$

where, recalling that $Y(t) = \mathbb{E}^{\mathbb{P}}_t[e^{\int_0^T e^{-\delta u} a \log X(u)\,\mathrm{d}u}]$,

$$\widehat{F}(t,\phi(t),\widehat{U}_2(t);\widehat{\mathbb{Q}}^*) := \begin{cases} e^{-\rho}e^{\frac{1-e^{-\delta T}}{\delta}a\log s^*}Y(t)\frac{C(t)^{1-\gamma}}{1-\gamma} - \delta\widehat{U}_2(t) & \text{for } 0\le t<T, \\ e^{-\rho}e^{\frac{1-e^{-\delta T}}{\delta}a\log s^*}Y(T)W(T) & \text{for } t=T. \end{cases}$$

Denote $\widehat{F}_u(t) := \frac{\partial\widehat{F}(t)}{\partial\widehat{U}_2(t)}$, $\widehat{F}_c(t) := \frac{\partial\widehat{F}(t)}{\partial C(t)}$, and $\widehat{F}_{cc}(t) := \frac{\partial^2\widehat{F}(t)}{\partial C(t)^2}$. $\widehat{F}$ is concave in $(\phi,\,\widehat{U}_2)$. Define

$$\Lambda(t) := \mathcal{E}(t)\widehat{F}_\phi(t) := \begin{cases} \mathcal{E}(t)\widehat{F}_c(t) & \text{for } t\in[0,T), \\ \mathcal{E}(T)e^{-\rho}e^{\frac{1-e^{-\delta T}}{\delta}a\log s^*}Y(T) & \text{for } t=T. \end{cases}$$

where $\mathcal{E}(t) := e^{-\delta t}$ and $\widehat{F}_c(t) = e^{-\rho}e^{\frac{1-e^{-\delta T}}{\delta}a\log s^*}Y(t)C(t)^{-\gamma}$.

Lemma 5.2 *Fix S^* defined by Eq.(4.11). For $(\phi,\beta)\in\Phi(S^*)$ and for any $h\in\mathcal{H}$ such that*

$(\phi + h, \beta') \in \Phi(S^*)$,

$$\widehat{U}_2(\phi + h, S^*) \leq \widehat{U}_2(\phi, S^*) + (\Lambda | h).$$

Proof: See appendix.

We obtain our first main result:

Proposition 5.1 *Under Assumption 5.1 and Assumption 5.2, for the optimal contract S^* defined by Eq.*(4.11), $\Pi = \Lambda = \mathcal{E}\widehat{F}_\phi$ *is a state price at:*

$$\begin{aligned} \phi(t) &= \begin{cases} C^*(t) & \textit{for } 0 \leq t < T, \\ 0 & \textit{for } t = T \end{cases} \\ \beta(t) &= \quad 0 \qquad\quad \textit{for } 0 \leq t \leq T \end{aligned}$$

where C^* is defined by Eq.(4.10).

Proof: See appendix.

Let us look at how this result is related to the market equilibrium obtained in Section 4. In the environment of Section 4.2, from Eq.(4.7), owing to the market-clearing condition,

$$\widehat{F}_c(t)\Big|_{C(t)=C^*(t)} = e^{-\rho} e^{\frac{1-e^{-\delta T}}{\delta} a \log s^*} Y(t) C^*(t)^{-\gamma} = J_w$$

Therefore, the result of Proposition 5.1 is equivalent to the market equilibrium obtained in Section 4.

We thus obtain a state price in the market equilibrium – call it an equilibrium state price:

$$\Pi(t) = \Lambda^*(t) := \mathcal{E}(t) F_c^*(t) \quad \text{for } 0 \leq t < T. \tag{5.2}$$

where $F_c^*(t) := \widehat{F}_c(t)\Big|_{C(t)=C^*(t)} = e^{-\rho} e^{\frac{1-e^{-\delta T}}{\delta} a \log s^*} (1 - s^*)^{-\gamma} Y(t) X(t)^{-\gamma}$. Also, $\Pi(T) = \mathcal{E}(T) F_\phi^*(T)$.

5.3 Riskless rate and the market prices of diffusive and jump risks in equilibrium

We examine a riskless rate and the market prices of both risks in equilibrium. From Eqs.(5.1),(5.2),

$$\begin{aligned} \frac{\mathrm{d}\Pi(t)}{\Pi(t_-)} &= -r(t)\,\mathrm{d}t - \eta(t)^\top \mathrm{d}B(t) - \xi(t)^\top \mathrm{d}M(t) \\ = \frac{\mathrm{d}\Lambda^*(t)}{\Lambda^*(t_-)} &= -\delta\,\mathrm{d}t + \frac{\mathrm{d}F_c^*(t)}{F_c^*(t_-)}, \qquad \text{with} \quad \Pi(0) = \Lambda(0) = 1. \end{aligned} \tag{5.3}$$

Now, solving $\frac{\mathrm{d}F_c^*(t)}{F_c^*(t_-)}$ in Eq.(5.3), we obtain our second main result:

Proposition 5.2 *In the environment of Proposition 5.1, the equilibrium riskless rate and the equilibrium market prices of diffusive and jump risks in the absence of moral hazard, denoted by* r^s, η^s, ξ^s *respectively, are constants:*

$$\begin{aligned} r^s &= \delta + \gamma\mu^G - \frac{\gamma(\gamma+1)}{2}\sum_{j=1}^{n}(\sigma_j^G)^2 + \sum_{i=1}^{m}\left\{1 - (1+z_i^G)^{-\gamma} - \gamma z_i^G\right\}\lambda_i, \qquad (5.4) \\ \eta_j^s &= \gamma\sigma_j^G \quad \textit{for } j = 1,\cdots,n, \\ \xi_i^s &= 1 - (1+z_i^G)^{-\gamma} \quad \textit{for } i = 1,\cdots,m. \end{aligned}$$

On the other hand, the equilibrium riskless rate and the equilibrium market prices of diffusive and jump risks in the presence of moral hazard are time-varying:

$$\begin{aligned} r(t) &= r^s + \left(\begin{array}{l} \left(\gamma a \frac{e^{-\delta t} - e^{-\delta T}}{\delta}\right)\sum_{j=1}^{n}(\sigma_j^G)^2 \\ -\sum_{i=1}^{m}\left(1-(1+z_i^G)^{-\gamma}\right)\left(1-(1+z_i^G)^{a\frac{e^{-\delta t}-e^{-\delta T}}{\delta}}\right)\lambda_i \end{array}\right) \\ &= \delta + \gamma\mu^G + \left(\gamma a \frac{e^{-\delta t} - e^{-\delta T}}{\delta} - \frac{\gamma(\gamma+1)}{2}\right)\sum_{j=1}^{n}(\sigma_j^G)^2 \\ &\quad + \sum_{i=1}^{m}\left\{\left(1-(1+z_i^G)^{-\gamma}\right)(1+z_i^G)^{a\frac{e^{-\delta t}-e^{-\delta T}}{\delta}} - \gamma z_i^G\right\}\lambda_i, \\ \eta_j(t) &= \eta_j^s - a\frac{e^{-\delta t} - e^{-\delta T}}{\delta}\sigma_j^G = \left(\gamma - a\frac{e^{-\delta t} - e^{-\delta T}}{\delta}\right)\sigma_j^G \textit{ for } j = 1,\cdots,n, \\ \xi_i(t) &= \xi_i^s + (1+z_i^G)^{-\gamma}\left(1-(1+z_i^G)^{a\frac{e^{-\delta t}-e^{-\delta T}}{\delta}}\right) = 1 - (1+z_i^G)^{-\gamma + a\frac{e^{-\delta t}-e^{-\delta T}}{\delta}} \textit{ for } i = 1,\cdots,m. \end{aligned}$$

Proof: From Eq.(5.2),

$$\frac{\mathrm{d}\Lambda^*(t)}{\Lambda^*(t_-)} = -\delta\,\mathrm{d}t + \frac{\mathrm{d}(X(t)^{-\gamma})}{X(t_-)^{-\gamma}} + \frac{\mathrm{d}Y(t)}{Y(t_-)} + \frac{\mathrm{d}Y(t)\,\mathrm{d}(X(t)^{-\gamma})}{Y(t_-)X(t_-)^{-\gamma}}. \tag{5.5}$$

By Itô's formula,

$$
\begin{aligned}
\frac{\mathrm{d}(X(t)^{-\gamma})}{X(t_-)^{-\gamma}} &= \left(-\gamma\mu^G + \frac{\gamma(\gamma+1)}{2}\sum_{j=1}^{n}(\sigma_j^G)^2 + \sum_{i=1}^{m}\left\{(1+z_i^G)^{-\gamma} - 1 + \gamma z_i^G\right\}\lambda_i\right)\mathrm{d}t \\
&\quad -\gamma\sum_{j=1}^{n}\sigma_j^G\,\mathrm{d}B_j(t) + \sum_{i=1}^{m}\left\{(1+z_i^G)^{-\gamma} - 1\right\}\mathrm{d}M_i(t). \qquad (5.6)
\end{aligned}
$$

For the sake of comparison, we look at the case of no moral hazard. Under the restriction of the stationary linear payment rule, $\frac{\mathrm{d}\left(C(t)^{-\gamma}\right)}{C(t_-)^{-\gamma}} = \frac{\mathrm{d}\left(X(t)^{-\gamma}\right)}{X(t_-)^{-\gamma}}$ in the market equilibrium. Accordingly, the first two terms on Eq.(5.5) correspond to the equilibrium state price in the absence of moral hazard, denoted by Λ^s:

$$
\begin{aligned}
\frac{\mathrm{d}\Lambda^s(t)}{\Lambda^s(t_-)} &= -\delta\,\mathrm{d}t + \frac{\mathrm{d}(X(t)^{-\gamma})}{X(t_-)^{-\gamma}} \\
&= -\delta\,\mathrm{d}t + \left(-\gamma\mu^G + \frac{\gamma(\gamma+1)}{2}\sum_{j=1}^{n}(\sigma_j^G)^2 + \sum_{i=1}^{m}\left\{(1+z_i^G)^{-\gamma} - 1 + \gamma z_i^G\right\}\lambda_i\right)\mathrm{d}t \\
&\quad -\gamma\sum_{j=1}^{n}\sigma_j^G\,\mathrm{d}B_j(t) - \sum_{i=1}^{m}\left\{1 - (1+z_i^G)^{-\gamma}\right\}\mathrm{d}M_i(t). \qquad (5.7)
\end{aligned}
$$

From Eq.(5.3),

$$
\begin{aligned}
r^s &= \delta + \gamma\mu^G - \frac{\gamma(\gamma+1)}{2}\sum_{j=1}^{n}(\sigma_j^G)^2 + \sum_{i=1}^{m}\left\{1 - (1+z_i^G)^{-\gamma} - \gamma z_i^G\right\}\lambda_i, \\
\eta_j^s &= \gamma\sigma_j^G \quad \text{for } j = 1,\cdots,n, \\
\xi_i^s &= 1 - (1+z_i^G)^{-\gamma} \quad \text{for } i = 1,\cdots,m.
\end{aligned}
$$

In other words, these represent the equilibrium pricing behavior driven only by the aggregate shock $\frac{\mathrm{d}\left(X(t)^{-\gamma}\right)}{X(t_-)^{-\gamma}}$, as in the standard general-equilibrium exchange economies.

On the other hand, in this model, the equilibrium pricing behavior is influenced by moral hazard. The third and fourth terms on Eq.(5.5) stand for the effect of the moral hazard on the dynamics of the equilibrium state price. Recall from Lemma 4.2 that

$$
\frac{\mathrm{d}Y(t)}{Y(t_-)} = a\frac{e^{-\delta t} - e^{-\delta T}}{\delta}\sum_{j=1}^{n}\sigma_j^G\,\mathrm{d}B_j(t) + \sum_{i=1}^{m}\{(1+z_i^G)^{a\frac{e^{-\delta t} - e^{-\delta T}}{\delta}} - 1\}\,\mathrm{d}M_i(t). \qquad (5.8)
$$

From Eq.(5.7) and Eq.(5.8),

$$\frac{\mathrm{d}\Lambda^*(t)}{\Lambda^*(t_-)} = \frac{\mathrm{d}\Lambda^s(t)}{\Lambda^s(t_-)} + \begin{pmatrix} -\left(\gamma a \frac{e^{-\delta t}-e^{-\delta T}}{\delta}\right)\sum_{j=1}^{n}(\sigma_j^G)^2 \\ +\sum_{i=1}^{m}\left((1+z_i^G)^{-\gamma}-1\right)\left((1+z_i^G)^{a\frac{e^{-\delta t}-e^{-\delta T}}{\delta}}-1\right)\lambda_i \end{pmatrix}\mathrm{d}t$$
$$+a\frac{e^{-\delta t}-e^{-\delta T}}{\delta}\sum_{j=1}^{n}\sigma_j^G\,\mathrm{d}B_j(t) + \sum_{i=1}^{m}(1+z_i^G)^{-\gamma}\left((1+z_i^G)^{a\frac{e^{-\delta t}-e^{-\delta T}}{\delta}}-1\right)\mathrm{d}M_i(t). \quad (5.9)$$

From Eq.(5.3) and Eq.(5.9), the market prices of diffusive and jump risks are:

$$\begin{aligned} \eta_j(t) &= \eta_j^s - a\frac{e^{-\delta t}-e^{-\delta T}}{\delta}\sigma_j^G \\ &= \left(\gamma - a\frac{e^{-\delta t}-e^{-\delta T}}{\delta}\right)\sigma_j^G \quad \text{for } j=1,\cdots,n, \\ \xi_i(t) &= \xi_i^s + (1+z_i^G)^{-\gamma}\left(1-(1+z_i^G)^{a\frac{e^{-\delta t}-e^{-\delta T}}{\delta}}\right) \\ &= 1-(1+z_i^G)^{-\gamma+a\frac{e^{-\delta t}-e^{-\delta T}}{\delta}} \quad \text{for } i=1,\cdots,m. \end{aligned}$$

Also, with regard to the riskless rate,

$$\begin{aligned} r(t) &= r^s + \begin{pmatrix} \left(\gamma a\frac{e^{-\delta t}-e^{-\delta T}}{\delta}\right)\sum_{j=1}^{n}(\sigma_j^G)^2 \\ -\sum_{i=1}^{m}\left(1-(1+z_i^G)^{-\gamma}\right)\left(1-(1+z_i^G)^{a\frac{e^{-\delta t}-e^{-\delta T}}{\delta}}\right)\lambda_i \end{pmatrix} \\ &= \delta + \gamma\mu^G + \left(\gamma a\frac{e^{-\delta t}-e^{-\delta T}}{\delta} - \frac{\gamma(\gamma+1)}{2}\right)\sum_{j=1}^{n}(\sigma_j^G)^2 \\ &\quad + \sum_{i=1}^{m}\left\{\left(1-(1+z_i^G)^{-\gamma}\right)(1+z_i^G)^{a\frac{e^{-\delta t}-e^{-\delta T}}{\delta}} - \gamma z_i^G\right\}\lambda_i. \end{aligned}$$

□

We call $(r(t) - r^s)$ a moral hazard premium on the equilibrium riskless rate:

$$\begin{aligned} r(t) - r^s &= \left(\gamma a\frac{e^{-\delta t}-e^{-\delta T}}{\delta}\right)\sum_{j=1}^{n}(\sigma_j^G)^2 \\ &\quad + \sum_{i=1}^{m}\left((1+z_i^G)^{-\gamma}-1\right)\left(1-(1+z_i^G)^{a\frac{e^{-\delta t}-e^{-\delta T}}{\delta}}\right)\lambda_i. \end{aligned} \quad (5.10)$$

The premium is a distortion of the riskless rate as compared to the one under the uncontrolled probability measure. Also,

$$\begin{aligned}
\eta_j(t) - \eta_j^s &= -a\frac{e^{-\delta t} - e^{-\delta T}}{\delta}\sigma_j^G \leq 0 \\
&\quad \text{with equality if } t = T \quad \text{for } j = 1, \cdots, n, \\
\xi_i(t) - \xi_i^s &= (1 + z_i^G)^{-\gamma}\Big(1 - (1 + z_i^G)^{a\frac{e^{-\delta t} - e^{-\delta T}}{\delta}}\Big) \geq 0 \\
&\quad \text{with equality if } t = T \quad \text{for } i = 1, \cdots, m.
\end{aligned}$$

are moral hazard-induced distortions of the market prices of diffusive and jump risks, respectively.

We draw three asset-pricing implications of moral hazard from Proposition 5.2. Firstly, from Eq.(5.5) and Eq.(5.7), the market prices of diffusive and jump risks are distorted, as compared to the case of no moral hazard, only through $\frac{\mathrm{d}Y(t)}{Y(t_-)}$. More essentially, inside of $Y(t)$, $e^{a \log X(t) \int_t^T e^{-\delta u}\, \mathrm{d}u} = X(t)^{\alpha(t)}$, where $\alpha(t) = a \int_t^T e^{-\delta u}\, \mathrm{d}u = a\frac{e^{-\delta t} - e^{-\delta T}}{\delta}$, plays a pivotal role in distorting the market prices of diffusive and jump risks. Recall that $Y(t)$ represents a time-t distortion of the probability measure caused by the endowment process X all over $[0, T]$. In parallel with $Y(t)$, $X(t)^{\alpha(t)}$ represents a time-t distortion of the probability measure caused by the time-t endowment. From Eq.(5.8), the distortion of the market price of diffusive risk is the diffusion coefficient of the stochastic differential of the distortion of the probability measure. While the market price of diffusive risk in the absence of moral hazard (i.e., $\gamma\sigma_j^G$ for $j = 1, \cdots, n$) is positive because the investor is risk-averse, the distortion of the market price of diffusive risk is negative because a positive (negative) diffusive shock improves (decreases) the probability measure. The distortion of the probability measure works as a hedge against the diffusive shock. Thus, the moral hazard reduces the market price of diffusive risk. However, as $t \to T$, the effect of the reduction is diminishing to zero because $\alpha(t) \downarrow 0$.

On the other hand, the distortion of the market price of jump risk consists of two parts: (1) the jump coefficient of the distorted probability measure and (2) the jump coefficient of the quadratic covariation of the distorted probability measure and $X(t)^{-\gamma}$. The distortion of the market price of jump risk is positive because a negative jump shock decreases the probability measure. Thus, the moral hazard amplifies the market price of jump risk, as compared to the one in the absence of moral hazard (i.e., $1 - (1 + z_i^G)^{-\gamma} < 0$ for $i = 1, \cdots, m$). However, again, as $t \to T$, the effect is diminishing because $\alpha(t) \downarrow 0$.

Secondly, the moral hazard premium is positive, since both terms are positive on the right-hand side of Eq.(5.10). The positivity of the premium holds true for either the regular risk or the rare-event risk. More specifically, with regard to the first term, since the moral hazard-induced distortion of the probability measure is a hedge against the diffusive shock, it reduces the diffusive-risk premium by $\left(\gamma a \frac{e^{-\delta t}-e^{-\delta T}}{\delta}\right) \sum_{j=1}^{n} (\sigma_j^G)^2$. Thus, it raises the riskless rate by the amount, i.e., the first term is positive. It is diminishing as $t \to T$, in parallel with the distortion of the market price of diffusive risk.

On the other hand, the second term represents the compensation for the market-valued jump risk distorted by moral hazard $(\xi_i(t) - \xi_i^s)\,\mathrm{d}M_i$ for all $i = 1, \cdots, m$. The distortion of the market-valued jump risk aggravates average profitability in the financial markets by its covariance (adjusted by the size of $(1 + z_i^G)^{-\gamma}$) with the original jump risk $\xi_i^s\,\mathrm{d}M_i$. The second term then makes up for the loss of the profitability. Since $-1 < z_i^G < 0$, $1 - (1 + z_i^G)^{-\gamma} < 0$ for all $i = 1, \cdots, m$. Thus, the second term is positive and raises the jump-risk premium as compared to the one in the absence of moral hazard (i.e., $\sum_{i=1}^{m} \left\{(1 + z_i^G)^{-\gamma} - 1\right\} \lambda_i$).[23] Again, it is diminishing as $t \to T$.

These results give new insights into the risk-free rate puzzle, which was explored first by Weil (1989). From Eq.(5.4), in the absence of moral hazard, large negative jumps (i.e., low z^G) tend to mitigate the puzzle by lowering the riskless rate r^s. However, from Eq.(5.10), the moral hazard raises the riskless rate. It implies that the risk-free puzzle is further exaggerated under moral hazard.

Thirdly, we examine the role of the financial markets in the moral hazard problem. For the sake of comparison, we derive the optimal sharing ratio in the absence of moral hazard, let it be denoted by s^{**}. Similarly to Eq.(4.4) above, the participation constraint binds under the linear payment rule. From Assumption 2.1, there exists $s^{**} \in (0, 1) \subset \mathbf{R}_{++}$ such that

$$\begin{aligned} \rho &= \frac{(1 - e^{-\delta T}) a \log s^{**}}{\delta} + \mathbb{E}^{\mathbb{P}}\left[\int_0^T e^{-\delta u} a \log X(u)\,\mathrm{d}u\right]. \\ \text{I.e.,} \quad s^{**} &= \exp\left\{ \begin{array}{l} -\log x_0 + \frac{\delta}{(1-e^{-\delta T})} \frac{\rho}{a} \\ -\frac{1-e^{-\delta T}(1+\delta T)}{\delta(1-e^{-\delta T})} \left(\begin{array}{l} \mu^G - \frac{1}{2}\sum_{j=1}^{n} (\sigma_j^G)^2 \\ + \sum_{i=1}^{m} \lambda_i \Big(\log(1 + z_i^G) - z_i^G \Big) \end{array} \right) \end{array} \right\}. \end{aligned} \tag{5.11}$$

[23]For positive jumps (i.e., $z_i^G > 0$ for some i), the second term could be negative. As the positive jumps are a reward for the investor, the negative return on the investment would be demanded.

Since $\mathbb{E}^{\mathbb{P}}\left[\int_0^T e^{-\delta u} a \log X(u)\,\mathrm{d}u\right] \leq \log \mathbb{E}^{\mathbb{P}}\left[e^{\int_0^T e^{-\delta u} a \log X(u)\,\mathrm{d}u}\right]$, from Eq.(4.5) and Eq.(5.11),

$$s^* \leq s^{**}. \tag{5.12}$$

This inequality may look counter-intuitive, in light of the standard moral hazard literature in corporate finance (e.g. Cvitanic and Zhang (2007)). From the literature, we might guess that the derivative $\frac{\mathrm{d}S}{\mathrm{d}X}$ would be higher under moral hazard than under no moral hazard, because the investor needs to give the firm an incentive to make higher efforts in the conflict. On the contrary, in our model, the sharing ratio is lower under moral hazard.

The reason for the inequality is that the markets alleviate the allocation conflict caused by moral hazard in our model. In the standard moral hazard models in which the investor does not have access to financial markets, the whole effect of moral hazard should be absorbed only in the allocation. On the other hand, by contrast, in our model, the effect is absorbed in the markets as well. More specifically, in the Lagrangian (4.1), the allocation is governed by the binding participation constraint, while the markets (and so the state prices) are governed by the other state constraint (i.e., the incentive constraint).[24] Thus, the markets play the role of relieving the allocation conflict.

5.4 Approximated effect of non-stationarity: Perturbation via the Taylor series expansion

We finally check the effect of non-stationarity of the linear contract. We perturb the above-obtained sharing ratio s^* via the Taylor series expansion with respect to time up to the second order. Note that we do not optimize s^* with respect to time.

The perturbed contract form is written as: letting $s^\zeta(t)$ denote the perturbed sharing ratio,

$$S(t) = s^\zeta(t) X(t) \in \mathcal{S} \quad \text{with} \quad s^\zeta(t) := \hat{s} e^{\nu t + \varepsilon t^2}$$

where $\hat{s}$ takes on either the value s^*, defined in Eq.(4.5), in the presence of moral hazard or the value s^{**}, defined in Eq.(5.11), in the absence of moral hazard. We impose the binding participation constraint, owing to the linearity assumption of the contract. Substituting s^ζ either into Eq.(4.5)

[24] Note that, if the linearity assumption of the contract is relaxed, then the allocation could be governed by the incentive constraint as well. The inequality relationship could then be reversed.

or Eq.(5.11),

$$\rho = \int_0^T e^{-\delta u} a\Big(\nu u + \varepsilon u^2\Big)\,\mathrm{d}u + \rho,$$

I.e.,

$$\begin{aligned}\nu &= -\frac{1}{\delta}\left(\frac{2 - e^{-\delta T}(2 + 2\delta T + \delta^2 T^2)}{1 - e^{-\delta T}(1 + \delta T)}\right)\varepsilon \\ &=: D(\delta, T)\varepsilon.\end{aligned}$$

We can thus choose such small values ε that $s^{\zeta}(t) \in (0, 1)$ for all t.

From Eq.(5.5), the effect of the perturbation is as follows. Since the perturbation is deterministic under the linear payment rule, the quadratic covariation is unchanged on Eq.(5.5). Hence, the market prices of diffusive and jump risks are unchanged by the perturbation. On the other hand, the drift term (i.e., the riskless rate) is influenced. The effect on the riskless rate is

$$\frac{\mathrm{d}(1 - s^{\zeta}(t))^{-\gamma}}{(1 - s^{\zeta}(t))^{-\gamma}}\frac{1}{\mathrm{d}t} = -\gamma \frac{\mathrm{d}\log(1 - s^{\zeta}(t))}{\mathrm{d}t} = \frac{\gamma \hat{s}}{1 - \hat{s}} D(\delta, T)\varepsilon.$$

Thus, the effect depends on δ and T. For T large enough, $D(\delta, T) = -\frac{2}{\delta} < 0$. Since $s^* \leq s^{**}$ from Eq.(5.12), for such T large enough, when $\varepsilon > 0$ ($\varepsilon < 0$), the effect is negative (positive), i.e., the moral hazard premium is lower (higher) under the perturbation than under the stationary linear payment rule.

6 Concluding remarks

This paper provided an explicit asset pricing formula under the moral hazard problem and obtained equilibrium state prices (in particular, the equilibrium riskless rate and the market prices of diffusive and jump risks). It then made clear the structural effect of moral hazard on equilibrium asset prices. Notably, under the moral hazard problem, a positive moral hazard premium was stipulated on the riskless rate in market equilibrium. The risk-free rate puzzle was further exaggerated in the presence of moral hazard. Also, the markets alleviated the allocation conflict caused by moral hazard.

However, there are two limitations in this paper. First, we assumed the CRRA/log utility function. It sure was more general than the exponential utility function, but was still time- and state-separable. It would be better to have more general utility forms, such as recursive utility, habit formation, and ambiguity aversion. The risk-free rate puzzle might then be, at least partly, resolved (see e.g. Nakamura et al. (2009)).

Second, we focused on the environment in which the linear contract was plausible. But, it would be desirable to look at more general environments in which optimal contracts should take on more general forms. For example, as we mentioned above, if the firm's instantaneous utility function is not of the logarithm type, then the linearity assumption of the contract is not plausible. More elaborate numerical work would then be required to find explicit solutions. Our explicit results obtained under the stationary linear contract would be useful as a benchmark for numerical approximations.

Appendix

A Proof of Lemma 3.1

Taking exponential of $\mathbb{E}^{\mathbb{Q}}[\int_0^T e^{-\delta u} a \log S(u)\,\mathrm{d}u] - H(\mathbb{Q}\,||\,\mathbb{P})$,

$$\begin{aligned}
e^{\mathbb{E}^{\mathbb{Q}}\left[\int_0^T e^{-\delta u} a \log S(u)\,\mathrm{d}u\right] - H(\mathbb{Q}\,||\,\mathbb{P})} &= e^{\mathbb{E}^{\mathbb{Q}}\left[\int_0^T e^{-\delta u} a \log S(u)\,\mathrm{d}u - \log\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right]} \\
&\leq \mathbb{E}^{\mathbb{Q}}\left[e^{\int_0^T e^{-\delta u} a \log S(u)\,\mathrm{d}u - \log\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}}\right] \quad \text{(by Jensen's inequality)} \\
&= \mathbb{E}^{\mathbb{Q}}\left[e^{\int_0^T e^{-\delta u} a \log S(u)\,\mathrm{d}u}\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}}\right] = \mathbb{E}^{\mathbb{P}}\left[e^{\int_0^T e^{-\delta u} a \log S(u)\,\mathrm{d}u}\mathbf{1}_{\{\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}>0\}}\right] \\
&\leq \mathbb{E}^{\mathbb{P}}\left[e^{\int_0^T e^{-\delta u} a \log S(u)\,\mathrm{d}u}\right]
\end{aligned}$$

with equality if and only if $\int_0^T e^{-\delta u} a \log S(u)\,\mathrm{d}u - \log\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}$ is a constant. Therefore,

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \frac{e^{\int_0^T e^{-\delta u} a \log S(u)\,\mathrm{d}u}}{\mathbb{E}^{\mathbb{P}}\left[e^{\int_0^T e^{-\delta u} a \log S(u)\,\mathrm{d}u}\right]}.$$

Thus, $\mathbb{Q}^*$ is obtained. □

B Proof of Lemma 4.1

From Eq.(2.5), $\mathrm{d}X(t) = X(t_-)\Big(\mu^G\,\mathrm{d}t + \sum_{j=1}^n \sigma_j^G\,\mathrm{d}B_j(t) + \sum_{i=1}^m z_i^G\,\mathrm{d}M_i(t)\Big)$, $X(0) = x_0 > 0$. Hence,

$$\begin{aligned} X(t) &= x_0 e^{\mu^G t + \sum_{j=1}^n \left(\sigma_j^G B_j(t) - \frac{(\sigma_j^G)^2}{2}t\right) - \sum_{i=1}^m z_i^G \lambda_i t} \prod_{i=1}^m \prod_{0\le s\le t} (1 + z_i^G \Delta N_i(s)), \\ \text{i.e., } \log\frac{X(t)}{x_0} &= \mu^G t + \sum_{j=1}^n \Big(\sigma_j^G B_j(t) - \frac{(\sigma_j^G)^2}{2}t\Big) - \sum_{i=1}^m z_i^G\lambda_i t + \sum_{i=1}^m \sum_{0\le s\le t} \log(1 + z_i^G \Delta N_i(s)) \\ &= \Big(\mu^G - \sum_{j=1}^n \frac{(\sigma_j^G)^2}{2} - \sum_{i=1}^m z_i^G\lambda_i\Big)t + \sum_{j=1}^n \sigma_j^G B_j(t) + \sum_{i=1}^m \Big(\log(1+z_i^G)\Big) N_i(t) \quad \text{(B.1)} \end{aligned}$$

Substituting Eq.(B.1) into Eq.(4.2), we examine each term on the right-hand side of Eq.(B.1). Firstly, the t-term is simply solved. Secondly, we look at the B_j-term ($j = 1, \cdots, n$). In particular, for a constant c, we solve $\mathbb{E}^{\mathbb{P}}[e^{c\int_0^T e^{-\delta t} B_j(t)\,\mathrm{d}t}]$. By Itô's formula,

$$-\frac{e^{-\delta T}}{\delta} B_j(T) = \int_0^T e^{-\delta t} B_j(t)\,\mathrm{d}t - \int_0^T \frac{e^{-\delta t}}{\delta}\,\mathrm{d}B_j(t) \quad (\because \mathrm{d}B_j(t)\,\mathrm{d}t = 0).$$

$$\begin{aligned} \text{I.e., } \int_0^T e^{-\delta t} B_j(t)\,\mathrm{d}t &= -\frac{e^{-\delta T}}{\delta} B_j(T) + \frac{1}{\delta}\int_0^T e^{-\delta t}\,\mathrm{d}B_j(t) \\ &= -\frac{e^{-\delta T}}{\delta}\int_0^T \mathrm{d}B_j(t) + \frac{1}{\delta}\int_0^T e^{-\delta t}\,\mathrm{d}B_j(t) \\ &= \frac{1}{\delta}\int_0^T \Big(e^{-\delta t} - e^{-\delta T}\Big)\,\mathrm{d}B_j(t) \qquad \sim \quad \mathcal{N}\Big(0, \frac{1}{\delta^2}\int_0^T \Big(e^{-\delta t} - e^{-\delta T}\Big)^2\,\mathrm{d}t\Big) \end{aligned}$$

where, for some $\mu \in \mathbf{R}$ and $\sigma > 0$, $\mathcal{N}(\mu, \sigma^2)$ denotes a normal distribution with mean μ and variance σ^2. Therefore,

$$\mathbb{E}^{\mathbb{P}}[e^{c\int_0^T e^{-\delta t} B_j(t)\,\mathrm{d}t}] = e^{\frac{c^2}{2}\frac{1}{\delta^2}\int_0^T \left(e^{-\delta t} - e^{-\delta T}\right)^2\,\mathrm{d}t}. \tag{B.2}$$

Lastly, we look at the N_i-term ($i = 1, \cdots, m$). In particular, for a constant c, we solve $\mathbb{E}^{\mathbb{P}}[e^{c\int_0^T e^{-\delta t} N_i(t)\,\mathrm{d}t}]$. Still, before solving it directly, we make preparations for it. Let the k-th jump time of N_i be denoted by τ_k. Consider the case of $N_i(T) = p$ for some integer $p \ge 1$. We can choose distinct $2p$ points of time $(s_1, s_2, \cdots, s_p)$ and $(t_1, t_2, \cdots, t_p)$ satisfying

$$0 \le s_1 < t_1 < s_2 < t_2 < \cdots < s_p < t_p < T.$$

We then take a joint probability conditional on $N_i(T) = p$ as follows:

$$\begin{aligned}
&\mathbb{P}[s_k < \tau_k < t_k,\, k = 1, \cdots p \,|\, N_i(T) = p\,] \\
=\;& \mathbb{P}[N_i(s_1) = 0,\, N_i(t_1) = 1,\, N_i(s_2) = 1,\, \cdots,\, N_i(s_p) = p-1,\, N_i(t_p) = p \,|\, N_i(T) = p\,] \\
=\;& \frac{1}{\frac{(\lambda_i T)^p}{p!} e^{-\lambda_i T}} \left(\begin{array}{c} e^{-\lambda_i s_1} \lambda_i (t_1 - s_1) e^{-\lambda_i (t_1 - s_1)} e^{-\lambda_i (s_2 - t_1)} \lambda_i (t_2 - s_2) e^{-\lambda_i (t_2 - s_2)} \\ \cdots e^{-\lambda_i (s_p - t_p)} \lambda_i (t_p - s_p) e^{-\lambda_i (T - s_p)} \end{array} \right) \\
=\;& p! \prod_{k=1}^{p} \left(\frac{t_k - s_k}{T} \right). \qquad \text{(B.3)}
\end{aligned}$$

As a brief remark, we give an intuitive explanation to the conditional joint probability. As $t_k \to s_k$ for all k, the corresponding joint probability density conditional on $N_i(T) = p$ is $\frac{p!}{T^p}$. As it is well known, the volume of the domain $\{(t_1, \cdots, t_p) \in \mathbf{R}^p \,|\, 0 < t_1 < t_2 < \cdots < t_p < T\}$ is $\frac{T^p}{p!}$. Hence, the asymptotic joint probability density is equivalent to a uniform distribution over the volume.

On the other hand, we take p independent real-valued random variables $U_1, U_2, \cdots, U_p$, each of which is defined on $[0, T]$ with a uniform distribution. For each scenario $\omega \in \Omega$, let the p realizations be arranged in ascending order as $R_1, R_2, \cdots, R_p$, i.e., $R_1 \leq R_2 \leq \cdots \leq R_p$. Note that the equalities hold with probability zero. Thus, applying Eq.(B.3) to this case,

$$\mathbb{P}[s_k < R_k < t_k,\, k = 1, \cdots p\,] = p! \prod_{k=1}^{p} \left(\frac{t_k - s_k}{T} \right).$$

Since $\mathbb{P}[s_k < R_k < t_k,\, k = 1, \cdots p\,] = p!\, \mathbb{P}[s_k < U_k < t_k,\, k = 1, \cdots p\,]$,

$$\mathbb{P}[s_k < U_k < t_k,\, k = 1, \cdots p\,] = \prod_{k=1}^{p} \left(\frac{t_k - s_k}{T} \right). \qquad \text{(B.4)}$$

Now, using Eq.(B.4) and the terminology used during its derivation, we solve $\mathbb{E}^{\mathbb{P}}[e^{c \int_0^T e^{-\delta t} N_i(t)\, \mathrm{d}t}]$ for a constant c. Taking expectation conditional on $N_i(T) = p$,

$$\mathbb{E}^{\mathbb{P}}[e^{c \int_0^T e^{-\delta t} N_i(t)\, \mathrm{d}t}] = \sum_{p=0}^{\infty} \mathbb{P}[\, N_i(T) = p\,]\, \mathbb{E}^{\mathbb{P}}[e^{c \int_0^T e^{-\delta t} N_i(t)\, \mathrm{d}t} \,|\, N_i(T) = p\,].$$

Focusing on $\int_0^T e^{-\delta t} N_i(t)\,\mathrm{d}t$ in the equation,

$$\begin{aligned}\int_0^T e^{-\delta t} N_i(t)\,\mathrm{d}t &= 0\cdot\frac{1-e^{-\delta\tau_1}}{\delta} + 1\cdot\frac{e^{-\delta\tau_1}-e^{-\delta\tau_2}}{\delta} + \cdots \\ &\quad +(p-1)\cdot\frac{e^{-\delta\tau_{p-1}}-e^{-\delta\tau_p}}{\delta} + p\cdot\frac{e^{-\delta\tau_p}-e^{-\delta T}}{\delta} \\ &= -p\frac{e^{-\delta T}}{\delta} + \sum_{k=1}^{p}\frac{e^{-\delta\tau_k}}{\delta} \\ &\overset{(d)}{=} -p\frac{e^{-\delta T}}{\delta} + \sum_{k=1}^{p}\frac{e^{-\delta R_k}}{\delta} = -p\frac{e^{-\delta T}}{\delta} + \sum_{k=1}^{p}\frac{e^{-\delta U_k}}{\delta}\end{aligned}$$

where the equality $\overset{(d)}{=}$ stands for equality in probability distribution. By Eq.(B.4),

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}[e^{c\int_0^T e^{-\delta t} N_i(t)\,\mathrm{d}t}] &= \sum_{p=0}^{\infty}\frac{(\lambda_i T)^p}{p!}e^{-\lambda_i T}\times e^{-pc\frac{e^{-\delta T}}{\delta}}\mathbb{E}^{\mathbb{P}}[e^{c\sum_{k=1}^{p}\frac{e^{-\delta U_k}}{\delta}}] \\ &= \sum_{p=0}^{\infty}\frac{(\lambda_i T)^p}{p!}e^{-\lambda_i T}\times e^{-pc\frac{e^{-\delta T}}{\delta}}\prod_{k=1}^{p}\mathbb{E}^{\mathbb{P}}[e^{c\frac{e^{-\delta U_k}}{\delta}}] \\ &= \sum_{p=0}^{\infty}\frac{(\lambda_i T)^p}{p!}e^{-\lambda_i T}\times e^{-pc\frac{e^{-\delta T}}{\delta}}\left(\frac{1}{T}\int_0^T e^{\frac{c}{\delta}e^{-\delta u}}\,\mathrm{d}u\right)^p \\ &= \exp\left\{\lambda_i T\left(-1+e^{-c\frac{e^{-\delta T}}{\delta}}\frac{1}{T}\int_0^T e^{\frac{c}{\delta}e^{-\delta u}}\,\mathrm{d}u\right)\right\}. \qquad \text{(B.5)}\end{aligned}$$

where $\int_0^T e^{\frac{c}{\delta}e^{-\delta u}}\,\mathrm{d}u$ is called an exponential integral.[25]

Finally, we solve $\mathbb{E}_t^{\mathbb{P}}\left[e^{\int_0^T e^{-\delta u} a\log X(u)\,\mathrm{d}u}\right]$ explicitly. From Eq.(B.1),

$$\begin{aligned}&\mathbb{E}_t^{\mathbb{P}}\left[e^{\int_0^T e^{-\delta u} a\log X(u)\,\mathrm{d}u}\right] \\ =\ & e^{a\int_0^t e^{-\delta u}\log X(u)\,\mathrm{d}u}\mathbb{E}\left[e^{ae^{-\delta t}\int_0^{T-t} e^{-\delta u} a\log\widetilde{X}(u)\,\mathrm{d}u}\,\middle|\,\widetilde{X}(0)=X(t)\right]\end{aligned}$$

where $\log\widetilde{X}(u) = \log X(0) + (\mu^G - \frac{1}{2}\sum_{j=1}^n(\sigma_j^G)^2 - \sum_{i=1}^m\lambda_i)u + \sum_{j=1}^n\sigma_j^G B_j(u) + \sum_{i=1}^m \log(1+$

[25]For the reference, we look at the case of $\delta=0$, although $\delta=0$ is not assumed in this paper. By taking the same procedures,

$$\mathbb{E}^{\mathbb{P}}[e^{c\int_0^T N_i(t)\,\mathrm{d}t}] = \exp\left\{\lambda_i T\left(-1+\frac{e^{cT}-1}{cT}\right)\right\}.$$

$z_i^G)N_i(u)$. Using Eq.(B.2) and Eq.(B.5),

$$
\begin{aligned}
&\mathbb{E}_t^{\mathbb{P}}\left[e^{\int_0^T e^{-\delta u} a \log X(u)\,\mathrm{d}u}\right] \\
= \; &\exp\begin{pmatrix}
a\int_0^t e^{-\delta u}\log X(u)\,\mathrm{d}u + a\frac{e^{-\delta t}-e^{-\delta T}}{\delta}\log X(t) \\
+ae^{-\delta t}(\mu^G - \frac{1}{2}\sum_{j=1}^n(\sigma_j^G)^2 - \sum_{i=1}^m \lambda_i)\int_0^{T-t} e^{-\delta u} u\,\mathrm{d}u \\
+\sum_{j=1}^n \frac{(ae^{-\delta t}\sigma_j^G)^2}{2}\frac{1}{\delta^2}\int_0^{T-t}(e^{-\delta u} - e^{-\delta(T-u)})^2\,\mathrm{d}u \\
+\sum_{i=1}^m \lambda_i(T-t)(-1+e^{-\frac{e^{-\delta(T-t)}}{\delta}\log(1+z_i^G)}\frac{1}{T-t}\int_0^{T-t} e^{\frac{e^{-\delta u}}{\delta}\log(1+z_i^G)}\,\mathrm{d}u)
\end{pmatrix}
\end{aligned}
$$

By integration by parts, $\int_0^{T-t} e^{-\delta u} u\,\mathrm{d}u = \frac{1-\exp\{-\delta(T-t)(1+\delta(T-t))\}}{\delta^2}$. □

C Proof of Lemma 4.2

Recalling Eq.(2.5),

$$\mathrm{d}X(t) = X(t_-)\Big(\mu^G\,\mathrm{d}t + \sum_{j=1}^n \sigma_j^G\,\mathrm{d}B_j(t) + \sum_{i=1}^m z_i^G\,\mathrm{d}M_i(t)\Big), \quad X(0) = x_0 > 0.$$

Thus,

$$\begin{cases} \mathrm{d}X^c(t) = X(t_-)\Big(\mu^G\,\mathrm{d}t + \sum_{j=1}^n \sigma_j^G\,\mathrm{d}B_j(t) - \sum_{i=1}^m z_i^G\lambda_i\,\mathrm{d}t\Big), \\ \Delta X(t) = X(t_-)\sum_{i=1}^m z_i^G \Delta N_i(t). \end{cases}$$

When a jump of N_i occurs at time t,

$$\Delta X(t) = X(t_-)z_i^G, \quad \text{i.e.,} \quad X(t) = X(t_-)(1+z_i^G). \tag{C.1}$$

For a twice continuously differentiable function $f(x,t)$, by Itô's formula,

$$
\begin{aligned}
f(X(t),t) \;=\; & f(X(0),0) + \int_0^t \frac{\partial f}{\partial x}(X(u_-),u)\,\mathrm{d}X^c(u) + \int_0^t \frac{\partial f}{\partial t}(X(u_-),u)\,\mathrm{d}u \\
& + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(X(u_-),u)(\mathrm{d}X^c(u))^2 + \sum_{0<u\le t}\Delta f(X(u),u).
\end{aligned}
$$

First, set $f(x,t) = \log x$. $\frac{\partial f}{\partial x} = \frac{1}{x}$, $\frac{\partial^2 f}{\partial x^2} = -\frac{1}{x^2}$, and $\frac{\partial f}{\partial t} = 0$. Taking logarithms on both sides of Eq.(C.1),

$$\log X(t) = \log X(t_-) + \log(1 + z_i^G) \quad \text{i.e.,} \quad \Delta \log X(t) = \log(1 + z_i^G).$$

Thus, for $u > t$,

$$\begin{aligned} \log X(u) &= \log X(t) + \sum_{j=1}^{n} \sigma_j^G (B_j(u) - B_j(t)) + \Big(\mu^G - \frac{1}{2}\sum_{j=1}^{n} (\sigma_j^G)^2 - \sum_{i=1}^{m} z_i^G \lambda_i\Big)(u - t) \\ &\quad + \sum_{i=1}^{m} \log(1 + z_i^G) \left\{N_i(u) - N_i(t)\right\}. \end{aligned}$$

Hence, $\log X(u) - \log X(t)$ is independent of $\mathcal{F}_t$. On the other hand,

$$\begin{aligned} Y(t) &= \mathbb{E}_t^{\mathbb{P}}\left[e^{a\int_0^T e^{-\delta u} \log X(u)\,\mathrm{d}u}\right] \\ &= e^{a\int_0^t e^{-\delta u} \log X(u)\,\mathrm{d}u}\, e^{a(\log X(t))\int_t^T e^{-\delta u}\,\mathrm{d}u}\, \mathbb{E}_t^{\mathbb{P}}\left[e^{a\int_t^T e^{-\delta u}\{\log X(u) - \log X(t)\}\,\mathrm{d}u}\right]. \end{aligned}$$

There, the second factor is $X(t)^{\alpha(t)}$, where $\alpha(t) := a\int_t^T e^{-\delta u}\,\mathrm{d}u = a\frac{e^{-\delta t} - e^{-\delta T}}{\delta}$. Also, the third factor is equivalent to the unconditional expected value, and thus is a deterministic function that is differentiable with respect to t. Therefore, the stochastic differential of $\widetilde{Y}(t) := \frac{Y(t)}{X(t)^{\alpha(t)}}$ takes the form of $\mathrm{d}\widetilde{Y}(t) = \cdots \mathrm{d}t$. That is, there exist neither $\mathrm{d}B_j(t)$-terms nor $\mathrm{d}N_i(t)$-terms in it.

Next, we look at the dynamics of $X(t)^{\alpha(t)}$. Set $f(x,t) = x^{\alpha(t)}$ and apply Itô's formula to it. Note that $\frac{\partial f}{\partial x} = \alpha(t)x^{\alpha(t)-1}$. Since, as shown above, $X(t) = X(t_-)(1 + z_i^G)$ when a jump of N_i occurs at time t,

$$X(t)^{\alpha(t)} = X(t_-)^{\alpha(t)}(1 + z_i^G)^{\alpha(t)}, \quad \text{i.e.,} \quad \Delta X(t)^{\alpha(t)} = X(t_-)^{\alpha(t)}\{(1 + z_i^G)^{\alpha(t)} - 1\}.$$

Hence,

$$\begin{aligned} \mathrm{d}\Big(X(t)^{\alpha(t)}\Big) &= \alpha(t)X(t_-)^{\alpha(t)-1}X(t_-)\Big(\sum_{j=1}^{n} \sigma_j^G\,\mathrm{d}B_j(t) + \cdots \mathrm{d}t\Big) \\ &\quad + \cdots \mathrm{d}t + \cdots \mathrm{d}t + X(t_-)^{\alpha(t)} \sum_{i=1}^{m} \{(1 + z_i^G)^{\alpha(t)} - 1\}\,\mathrm{d}N_i(t). \end{aligned}$$

Therefore,

$$
\begin{aligned}
\mathrm{d}Y(t) &= \mathrm{d}\left(\widetilde{Y}(t)X(t)^{\alpha(t)}\right) \\
&= \widetilde{Y}(t_-)\,\mathrm{d}\left(X(t)^{\alpha(t)}\right) + X(t_-)^{\alpha(t)}\,\mathrm{d}\widetilde{Y}(t) + \mathrm{d}\widetilde{Y}(t)\,\mathrm{d}\left(X(t)^{\alpha(t)}\right) \\
&= \widetilde{Y}(t_-)X(t_-)^{\alpha(t)}\Big(\alpha(t)\sum_{j=1}^{n}\sigma_j^G\,\mathrm{d}B_j(t) + \sum_{i=1}^{m}\{(1+z_i^G)^{\alpha(t)}-1\}\,\mathrm{d}M_i(t) + \cdots\mathrm{d}t\Big) \\
&\quad + \cdots\mathrm{d}t + 0 \\
&= Y(t_-)\Big(\alpha(t)\sum_{j=1}^{n}\sigma_j^G\,\mathrm{d}B_j(t) + \sum_{i=1}^{m}\{(1+z_i^G)^{\alpha(t)}-1\}\,\mathrm{d}M_i(t)\Big) + \cdots\mathrm{d}t.
\end{aligned}
$$

Since Y is a martingale, the total sum of the dt-terms is zero. The result is obtained. □

D Proof of Lemma 5.1

For $(\phi,\beta)\in\Phi(S^*)$ satisfying Assumption 5.2, because of Assumption 5.1, by Lebesgue's dominated convergence theorem,

$$
\begin{aligned}
W(t) &= \mathbb{E}_t^{\mathbb{P}}\left[\int_t^T\left(\begin{array}{l}\frac{\Pi(s)}{\Pi(t)}\Big(C(s)+S^*(s)-X(s)\Big) \\ +\frac{\Pi(s)}{\Pi(t)}W(s)\beta(s)^\top\Big(\mu^R-\sigma^R\eta(s)-z^RI^\xi(s)\lambda\Big)\end{array}\right)\mathrm{d}s + \frac{\Pi(T)}{\Pi(t)}W(T)\right] \\
&= \mathbb{E}_t^{\mathbb{P}}\left[\int_t^T\frac{\Pi(s)}{\Pi(t)}\Big(C(s)+S^*(s)-X(s)\Big)\,\mathrm{d}s + \frac{\Pi(T)}{\Pi(t)}W(T)\right].
\end{aligned}
$$

$$
\begin{aligned}
\text{I.e., } w_0 &= \Pi(0)W(0) \\
&= \mathbb{E}^{\mathbb{P}}\left[\int_0^T\Pi(u)(C(u)+S^*(u)-X(u))\,\mathrm{d}u + \Pi(T)W(T)\right] = (\Pi|\phi+S^*-X).
\end{aligned}
$$

On the other hand, for any $h\in\mathcal{H}$ such that $\phi+h\in\Phi(S^*)$, by Fatou's lemma,

$$
\Pi(0)W(0) + (\Pi|X-S^*) \geq (\Pi|\phi+h).
$$

Therefore, $(\Pi|h)\leq 0$. □

E Proof of Lemma 5.2

Define

$$\begin{aligned}\Delta_U &:= \widehat{U}_2(\phi+h,S^*)-\widehat{U}_2(\phi,S^*),\\ \Delta_\Sigma &:= \Sigma(\phi+h,S^*)-\Sigma(\phi,S^*),\\ \Delta_\Gamma &:= \Gamma(\phi+h,S^*)-\Gamma(\phi,S^*).\end{aligned}$$

Also, define

$$\Delta_F(t) := \begin{cases} \widehat{F}(\phi,S^*,\widehat{U}_2)+\widehat{F}_c h-\delta\Delta_U-\widehat{F}(\phi+h,S^*,\widehat{U}_2+\Delta_U) & \text{for } 0\le t<T,\\ 0 & \text{for } t=T.\end{cases}$$

Due to the concavity of $\widehat{F}$, $\Delta_F(t)\ge 0\,\forall\, t$. Recalling $\mathrm{d}\widehat{U}_2=-\widehat{F}(\phi,S^*,\widehat{U}_2)\,\mathrm{d}t+\Sigma\,\mathrm{d}B+\Gamma\,\mathrm{d}M$; $\widehat{U}_2(T)=\frac{\mathrm{d}\widehat{\mathbb{Q}}^*}{\mathrm{d}\mathbb{P}}W(T)$,

$$\begin{aligned}\mathrm{d}\Delta_U &= \mathrm{d}\widehat{U}_2(\phi+h,S^*)-\mathrm{d}\widehat{U}_2(\phi,S^*)\\ &= -(\widehat{F}_c h-\delta\Delta_U-\Delta_F)\,\mathrm{d}t+\Delta_\Sigma\,\mathrm{d}B+\Delta_\Gamma\,\mathrm{d}M;\\ \Delta_U(T) &= \widehat{F}_\phi(T)h(T)-\Delta_F(T)=\widehat{F}_\phi(T)h(T).\end{aligned}$$

Since $\mathcal{E}$ is deterministic (i.e., $\mathrm{d}\mathcal{E}(t)=-\delta\,\mathrm{d}t$),

$$\begin{aligned}\mathrm{d}(\mathcal{E}\Delta_U) &= (-\mathcal{E}\widehat{F}_c h+\mathcal{E}\Delta_F)\,\mathrm{d}t+\cdots\mathrm{d}B+\cdots\mathrm{d}M;\\ &\mathcal{E}(T)\Delta_U(T)=\mathcal{E}(T)\widehat{F}_\phi(T)h(T)-\mathcal{E}(T)\Delta_F(T)=\mathcal{E}(T)\widehat{F}_\phi(T)h(T).\end{aligned}$$

By Lebesgue's dominated convergence theorem,

$$\begin{aligned}\Delta_U(0) &= \mathbb{E}^{\mathbb{P}}\left[\int_0^T(\mathcal{E}\widehat{F}_c h-\mathcal{E}\Delta_F)\,\mathrm{d}t+\Big(\mathcal{E}(T)\widehat{F}_\phi(T)h(T)-\mathcal{E}(T)\Delta_F(T)\Big)\right]\\ &= (\mathcal{E}\widehat{F}_\phi|h)-(\mathcal{E}|\Delta_F).\end{aligned}$$

Since $\Delta_F(t)\ge 0\,\forall\, t$, $\widehat{U}_2(\phi+h,S^*)-\widehat{U}_2(\phi,S^*)\le(\Lambda|h)$. □

F Proof of Proposition 5.1

By Lemma 5.1 and Lemma 5.2, for $(\phi, \beta) \in \mathcal{C}(S^*)$ satisfying Assumption 5.2 and for any $h \in \mathcal{H}$ such that $(\phi + h, \beta') \in \mathcal{C}(S^*)$,

$$\Delta_U(0) = \widehat{U}_2(\phi + h, S^*) - \widehat{U}_2(\phi, S^*) \leq (\Pi|h) \leq 0.$$

Hence, at such (ϕ, β), Π is a state price and the utility is maximized. Since the optimal C is uniquely determined for the given s^* in the market equilibrium where $W(t) = 0$ for all t, the equilibrium C is equal to C^* defined in Eq.(4.10). Obviously, such (ϕ, β) is in $\mathcal{C}(S^*)$. □

References

[1] Breeden, D., 1979, "An Intertemporal Asset Pricing Model with Stochastic Consumption and Investment Opportunities," *Journal of Financial Economics*, 7, 265-296.

[2] Brémaud, P., 1981, *Point Processes and Queues: Martingale Dynamics*, Springer.

[3] Cover, T.M., and Thomas, J.A., 2006, *Elements of Information Theory*, Wiley, 2nd Edition.

[4] Cox, J., Ingersoll, J., and Ross, S., 1985, "An Intertemporal General Equilibrium Model of Asset Prices," *Econometrica*, 53, 363-384.

[5] Cvitanić, J., Wan, X., and Zhang, J., 2009, "Optimal Compensation with Hidden Action and Lump-Sum Payment in a Continuous-Time Model," *Applied Mathematics and Optimization*, 59, 99-146.

[6] Cvitanić, J., and Zhang, J., 2007, "Optimal Compensation with Adverse Selection and Dynamic Actions," *Mathematics and Financial Economics*, 1, 21-55.

[7] Cvitanić, J., and Zhang, J., 2013, *Contract Theory in Continuous-Time Models*, Springer.

[8] Dana, R.A., and Jeanblanc, M., 2007, *Financial Markets in Continuous Time*, Springer.

[9] Delbaen, F., Grandits, P., Rheinlander, T., Samperi, D.J., Schweizer, M., and Stricker, C., 2002, "Exponential Hedging and Entropic Penalties," *Mathematical Finance*, 12, 99-123.

[10] Hansen, L.P., and Sargent, T.J., 2007, *Robustness*, Princeton University Press.

[11] Hansen, L.P., Sargent, T.J., Turmuhambetova, G., and Williams, N., 2006, "Robust Control and Model Misspecification," *Journal of Economic Theory*, 128, 45-90.

[12] Holmström, B., and Milgrom, P., 1987, "Aggregation and Linearity in the Provision of Intertemporal Incentives," *Econometrica*, 55, 303-328.

[13] Lucas, R., 1978, "Asset Prices in an Exchange Economy," *Econometrica*, 46, 1429-1445.

[14] Medvegyev, P., 2007, *Stochastic Integration Theory*, Oxford University Press.

[15] Misumi, T., Nakamura, H., and Takaoka, K., 2013, "Optimal Risk Sharing in the Presence of Moral Hazard under Market Risk and Jump Risk," HCFR Working Paper G-1-4.

[16] Nakamura, H., 2015, "Moral Hazard Premium under Diffusive Risk in the Strong Formulation," mimeo.

[17] Nakamura, H., Nozawa, W., and Takahashi, A., 2009, "Macroeconomic Implications of Term Structures of Interest Rates under Stochastic Differential Utility with non-unitary IES," Asia-Pacific Financial Markets, 16, 231-263.

[18] Nakamura, H., and Takaoka, K., 2013, "A Continuous-Time Optimal Insurance Design with Costly Monitoring," Working Paper, #140, Graduate School of Commerce and Management, Hitotsubashi University.

[19] Øksendal, B., and Sulem, A., 2007, *Applied Stochastic Control of Jump Diffusions*, Springer, 2nd Edition.

[20] Ou-Yang, H., 2005, "An Equilibrium Model of Asset Pricing and Moral Hazard," *Review of Financial Studies*, 18, 1253-1303.

[21] Pham, H., 2009, *Continuous-time Stochastic Control and Optimization with Financial Applications*, Springer.

[22] Protter, P.E., 2010, *Stochastic Integration and Differential Equations*, Version 2.1, Springer.

[23] Sannikov, Y., 2008, "A Continuous-Time Version of Principal-Agent Problem," *Review of Economic Studies*, 75, 957-984.

[24] Schättler, H., and Sung, J., 1993, "The First-Order Approach to the Continuous-Time Principal-Agent Problem with Exponential Utility," *Journal of Economic Theory*, 61, 331-371.

[25] Shreve, S.E., 2004, *Stochastic Calculus for Finance II: Continuous-Time Models*, Springer.

[26] Sims, C., 2003, "Implications of Rational Inattention," *Journal of Monetary Economics*, 50, 665-690.

[27] Skiadas, C., 2007, "Dynamic Portfolio Choice and Risk Aversion," in *Handbooks in Operations Research and Management Science: Financial Engineering*, Eds. by Birge, J.R., and Linetsky, V., Vol. 15, Ch. 19, 789-843, North-Holland.

[28] Tirole, J., 2006, *The Theory of Corporate Finance*, Princeton University Press.

[29] Yong, J., and Zhou, X.Y., 1999, *Stochastic Controls: Hamiltonian Systems and HJB Equations*, Springer.

[30] Weil, P., 1989, "The Equity Premium Puzzle and the Risk-Free Rate Puzzle," *Journal of Monetary Economics*, 24, 401-421.