Optimal Investment and Liability Ratio Policies in a Multidimensional Regime Switching Model

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Abstract

We consider an insurer who faces an external jump-diffusion risk that is negatively correlated with the capital returns in a multidimensional regime switching model. The insurer selects investment and liability ratio policies continuously to maximize her/his expected utility of terminal wealth. We obtain explicit solutions of optimal investment and liability ratio policies for logarithmic, power, and exponential utility functions. We study the impact of the insurer's risk attitude, the negative correlation between external risk and capital returns, and the regime of the economy, on optimal policy.

Key Words: jump diffusion; Markov chain; risk management; optimal investment/insurance; stochastic control; utility maximization.

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1 Introduction

1.1 Economic Motivations

The 2007-2009 financial crisis and economic recession almost put the global financial system on the brink of collapse. With the accurate total cost of the financial crisis being incalculable, economists have estimated a conservative number of the loss in the U.S. between $6 trillion and $14 trillion, or equivalently, $50,000 to $120,000 for every U.S. household (See Atkinson et al. (2013)). To further illustrate how severe this financial crisis was, we review the staggering case of American International Group, Inc. (AIG), once one of the largest and most successful insurance companies in the world. AIG's stock price was traded at over $50 per share in February 2008 before the financial crisis, but dropped to less than $2 per share in September 2008 when AIG was deep in the crisis. To prevent the financial system from breakdown, the U.S. government took over AIG through an initial rescue of $85 billion in September 2008, the largest bailout amount in the U.S. history. According to record, the total amount of rescue in the AIG case is over $182 billion (See Sjostrom (2009) for additional information).

Apart from the huge impact on the industry and the financial markets, this financial crisis also brought intense discussions and research to the academic field. Many academics investigate the influence of complicated financial products, such as credit default swaps (CDS), on the economy and the financial system. They debate over monetary policies, government intervention, regulation of the markets, systemic risk, et cétera. However, there are still many important open problems on quantitative investment and risk management with regards to the financial crisis.

1.2 Review of Consumption-Investment Models and Reinsurance Models

The study on consumption-investment problems in continuous-time started with the seminal paper of Merton (1969) in which dynamic programming was applied to obtain explicit optimal consumption and investment policies. Karatzas et al. (1986) provided a more general and rigorous analysis to Merton's problem, including arguments on whether the positive constraint of consumption is active, different scenarios for the natural payments, and conditions under which the value function is finite. Many early contributions to consumption-investment problems can be found in the monographs of Karatzas and Shreve (1998) and Sethi (1997). Browne (1995) further extended Merton's framework by assuming that investors are subject to an external risk process (modeled by a diffusion process), and found optimal investment policies under two criteria: maximizing exponential utility and minimizing the probability of ruin. Following the same vein, Yang and Zhang (2005) and Wang et al. (2007) used a jump-diffusion process to model investors' external risk. They both obtained explicit optimal investment strategies, but used different methods: the former by solving the associated HJB equations and the latter through the martingale approach. In optimal investment problems with an external risk process, e.g., above mentioned Browne (1995), Wang et al. (2007), and Yang and Zhang (2005), investors leave their external risk processes uncontrolled, which means the risk process is independent of investment decisions. Moore and Young (2006) incorporated an external risk (which can be insured against through the purchase of insurance policies) into Merton's model and considered consumption, investment and insurance problems, which was generalized in a regime switching model by Zou and Cadenillas (2014b).

In the insurance industry, a commonly used risk management tool for insurers is reinsurance. In a typical reinsurance problem, an insurer manages its risk exposure by controlling reinsurance strategies under certain objectives. A classical risk model in actuarial science is compound Poisson process, also called Cramér-Lungberg Model (e.g., Schmidli (2001)). Since the limit process of a compound Poisson process is a diffusion process, diffusion processes are also frequently used to model risk, see, for instance, Taksar (2000) and Hojgaard and Taksar (1998). Common reinsurance types used in the literature are proportional reinsurance (see Hojgaard and Taksar (1998)), and stop-loss reinsurance (see Kaluszka (2001)). Academics also study reinsurance problems under various objectives, such as mean-variance criterion in Kaluszka (2001), maximizing expected utility of running reserve in Hojgaard and Taksar (1998), minimizing the probability of ruin in Schmidli (2001), and maximizing expected utility of terminal wealth in Liu et al. (2013).

1.3 Review of the AIG Case

As pointed out in Stein (2012, Chapter 6), one major mistake in the AIG case was to ignore the negative correlation between its liabilities and the capital returns. Such correlation has also been ignored in consumption/investment problems with external risk process, and reinsurance problems with investment. To overcome this drawback, Stein (2012, Chapter 6) proposed a diffusion model for AIG's risk process that is negatively correlated with the stock price process. He then found the optimal liability ratio for AIG when its objective is to maximize the expected logarithmic utility of terminal wealth. Zou and Cadenillas (2014a) improved that model by including investment as a control. They obtained optimal investment and liability ratio strategies under HARA, CARA, and quadratic utility functions.

As agreed by most economists, the trigger of the 2007-2009 financial crisis was the crash of the housing market. But back at that time, most individual investors, companies, financial institutions and banks did not seriously consider the business cycles in the U.S. housing market and made their financial decisions based on the false prediction of the housing price index. In the AIG case, AIG Financial Products Corp. (AIGFP), AIG's subsidiary, significantly underestimated the risk of writing CDS backed by mortgage payments. To manage the risk generated by business cycles, regime switching models should be considered. See, for instance, Sotomayor and Cadenillas (2009), Zhou and Yin (2004), Zou and Cadenillas (2014b), and Bauerle and Rieder (2004), for references on regime switching models.

1.4 Contributions

Motivated by the infamous AIG case mentioned above, we propose a regime switching model which addresses two major mistakes AIG made in the financial crisis. We consider an insurer whose external risk (liabilities) is modeled by a jump-diffusion process and suppose the insurer can control the risk process. We assume the insurer makes investment decisions in a financial market which consists of a riskless asset and a finite number of risky assets. We also assume the insurer's risk process is negatively correlated with the price processes of the risky assets. In our model, both the financial market and the risk process depend on the regime of the economy. The objective of the insurer is to select the proportions of wealth invested in the risky assets and the liability ratio (which is defined as total liabilities over wealth) to maximize her/his expected utility of terminal wealth.

As far as we know, this is the first paper studying both investment and liability ratio problems when there is regime switching in the economy. We successfully obtain optimal investment and liability ratio policies in explicit forms for logarithmic utility, power utility and exponential utility. Stein

(2012, Chapter 6) considered a similar problem but in a much simpler framework than ours. First, he did not consider regime switching in the model. Second, the insurer did not control investment decisions, and the risk was modeled by a diffusion process without jumps. Last, the only utility function considered in Stein (2012, Chapter 6) was the logarithmic utility function. Zou and Cadenillas (2014a) studied the same problem but with only one risky asset in the financial market, and no regime switching in the economy. We have discussed the importance of incorporating regime switching into the model. Here, we consider a financial market of K risky assets, and each one of them has a different negative correlation with the external risk. We allow the regime of the economy to affect both the financial market and the risk process. Credit default swaps (CDS) give examples of risk processes strongly affected by the regime of the economy (see Harrington (2009) for a description of CDS). Different from consumption/investment models with regime switching like Bauerle and Rieder (2004), Sotomayor and Cadenillas (2009), and Zhou and Yin (2004), our model also incorporates an external risk process. Our research also differs from recent work in reinsurance problems in several directions. For instance, in Liu et al. (2013), the insurer's risk process is governed by a continuous diffusion process (without jumps) and is assumed to be independent of the price process of the securities. In Zhuo et al. (2013), investment is not included and they only provide numerical solutions.

This paper is organized as follows. In Section 2, we describe the regime switching model and formulate the problem. In Section 3, we develop the associated Hamilton-Jacobi-Bellman equation and provide the corresponding verification theorem. In Section 4, we obtain explicit solutions of optimal investment and liability ratio policies for logarithmic utility, power utility, and exponential utility functions. In Section 5, we present an economic analysis. Section 6 concludes our work.

2 The Model

We consider a continuous-time financial market model with regime switching. The regime of the economy is represented by an observable, continuous-time and stationary Markov chain $\epsilon = \{\epsilon_t, 0 \leq t \leq T\}$ with finite state space $\mathcal{S} = \{1, 2, \ldots, S\}$. Here $T \in (0, \infty)$ is the terminal time and $S \in \mathbb{N}^+$ is the number of regimes in the economy. We assume the Markov Chain ϵ has a strongly irreducible generator $Q = (q_{ij})_{S \times S}$, where $\sum_{j \in \mathcal{S}} q_{ij} = 0$ for every

$i \in \mathcal{S}$.

In the financial market, there exist one riskless asset (bond) and K risky assets (stocks). The price processes of the riskless asset and the risky assets are represented by P^0 and P^m, $m \in \mathcal{K} := \{1, 2, \cdots, K\}$, respectively, which satisfy the Markov-modulated stochastic differential equations:

$$\begin{aligned} dP^0(t) &= r(\epsilon_t)P^0(t)dt, \; P^0(0) = 1; \\ dP^m(t) &= P^m(t)\left(\mu^m(\epsilon_t)dt + \sum_{n=1}^{K} \sigma^{mn}(\epsilon_t)dW^{(n)}(t)\right), \; P^m(0) > 0. \end{aligned}$$

Here $W^{(n)}$, $n \in \mathcal{K}$, is a standard one-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $W^{(m)}$ is independent of $W^{(n)}$ if $m \neq n$. For every $\epsilon_t = i \in \mathcal{S}$, the coefficients[1] $r_i := r(\epsilon_t)$, $\mu_i^m := \mu^m(\epsilon_t)$, $\sigma_i^{mn} := \sigma^{mn}(\epsilon_t)$ are constants for every $m, n \in \mathcal{K}$. Furthermore, we denote the $K \times K$ matrix $\boldsymbol{\sigma}(\epsilon_t) := (\sigma^{mn}(\epsilon_t))$, and assume $\boldsymbol{\sigma}(\epsilon_t)$ is positive definite and invertible for every $t \in [0, T]$. We introduce the following vector notations: $\boldsymbol{W} = (W^{(1)}, W^{(2)}, \ldots, W^{(K)})$, $\boldsymbol{\mu} = (\mu^1, \mu^2, \ldots, \mu^K)$, $\mathbf{1} = (1, 1, \ldots, 1)$ (K-dimensional row vector of all ones), and $\boldsymbol{\Sigma}(\epsilon_t) = \boldsymbol{\sigma}(\epsilon_t)\boldsymbol{\sigma'}(\epsilon_t)$, where M' denotes transpose of a matrix M. Since $\boldsymbol{\sigma}^{-1}(\epsilon_t)$ exists, $\boldsymbol{\Sigma}^{-1}(\epsilon_t)$ also exists for every $t \in [0, T]$.

We consider an insurer who makes her/his investment decisions in the financial market by selecting the proportion of their wealth invested in each asset. We denote an investment policy by a K-dimensional process $\boldsymbol{\pi} := (\pi^1(t), \pi^2(t), \cdots, \pi^K(t))_{t \in [0,T]}$, where $\pi^m(t)$ is the proportion of wealth invested in m-th risky asset at time t. Hence, the proportion invested in the riskless asset at time t is $1 - \sum_{m=1}^{K} \pi^m(t)$. Here, we assume $\pi^m \in \mathbb{R}$ for each $m \in \mathcal{K}$, which means that we allow short-selling in the market.

We assume that the insurer sells insurance policies at a unit premium (per dollar amount of liabilities), given by $p(\epsilon_t)$ at time t, where $p_i > 0$ for every $i \in \mathcal{S}$. In the meantime, insurers are subject to the risk (liabilities) from the written insurance policies. Generalizing Wang et al. (2007) by allowing regime switching, we assume the unit risk (risk per dollar amount of liabilities) is modeled by a jump-diffusion process

$$dR(t) = a(\epsilon_t)dt + b(\epsilon_t)d\tilde{W}(t) + \gamma(\epsilon_t)dN(t),$$

where $\tilde{W}$ is a standard one-dimensional Brownian motion and N is a Poisson process with constant intensity $\lambda > 0$. For every $\epsilon_t = i \in \mathcal{S}$, the coefficients

[1] We use subscript i to denote the dependence on the regime ϵ_t when $\epsilon_t = i$.

a_i, b_i and γ_i are positive constants. Therefore, in our setting, insurers' unit profit (loss if being negative) over the time period $(t, t+dt)$ is $p(\epsilon_t)dt - dR(t)$. Our risk model allows not only traditional insurance, but also non-traditional insurance in which the risk is strongly affected by the regime of the economy (like CDS).

As proposed by Stein (2012, Chapter 6), the risk process R is negatively correlated with the capital gains in the financial market. We assume such negative correlation is captured by

$$d < \tilde{W}, W^{(n)} >_t = \rho^n(\epsilon_t)dt,$$

where $-1 \leq \rho_i^n < 0$ for every $\epsilon_t = i \in \mathcal{S}$.

We denote $\boldsymbol{\rho} = (\rho^1, \rho^2, \ldots, \rho^K)$. We also denote $|\boldsymbol{x}|^2 := \boldsymbol{x} \cdot \boldsymbol{x}'$, where $\boldsymbol{x}$ is a row vector. Then we obtain

$$d\tilde{W}(t) = \boldsymbol{\rho}(\epsilon_t) \cdot d\boldsymbol{W}'(t) + \sqrt{1 - |\boldsymbol{\rho}(\epsilon_t)|^2} \cdot d\bar{W}(t),$$

where $\bar{W}$ is a standard Brownian motion defined on $(\Omega, \mathcal{F}, \mathbb{P})$, and is independent of $W^{(n)}$ for every $n \in \mathcal{K}$.

In the insurance market, we assume insurers can control their total liabilities at time t, denoted by $L(t)$. Then the dynamics of the insurer's total profit is given by

$$L(t)\left(p(\epsilon_t)dt - dR(t)\right).$$

Following Sotomayor and Cadenillas (2009), we assume the Brownian motions $W^{(n)}$, $n \in \mathcal{K}$, and $\bar{W}$, the Poisson process N, and the Markov chain ϵ are mutually independent. We take the $\mathbb{P}-$augmented filtration generated by $W^{(n)}, \bar{W}, N$ and ϵ as our filtration $\{\mathcal{F}_t\}_{0 \leq t \leq T}$.

Remark 2.1 *The above model for risk process can be understood as a limiting process of the classical Cramér-Lundberg model, see, e.g., Taksar (2000), Wang et al. (2007), and Zhuo et al. (2013).*

Remark 2.2 *We assume the coefficients satisfy $\mu_i > r_i > 0$ and $p_i > a_i > 0$ for every $i \in \mathcal{S}$. Such assumption is reasonable, and is also in accordance with the financial markets in real life. This is due to the well accepted conclusion that extra uncertainty must be compensated by extra return.*

At time t, the insurer selects her/his investment policy $\boldsymbol{\pi}(t)$ and her/his liability ratio $\kappa(t)$, defined as the ratio of total liabilities over wealth. We define the control $u := \{(\boldsymbol{\pi}(t), \kappa(t))\}_{t\in[0,T]}$. For any control u, we denote $X^u(t)$ as the insurer's wealth (surplus) at time t, and thus $\kappa(t) = \frac{L(t)}{X^u(t)}$, where $L(t)$ represents the total liabilities at time t. Based on the above model setting, we obtain the dynamics of $X^u(t)$

$$\begin{aligned}\frac{dX^u(t)}{X^u(t-)} &= \big(\boldsymbol{\mu}(\epsilon_t) - r(\epsilon_t)\mathbf{1}\big) \cdot \boldsymbol{\pi}'(t)dt + \big[r(\epsilon_t) + (p(\epsilon_t) - a(\epsilon_t))\kappa(t)\big]dt \\ &\quad + \big[\boldsymbol{\pi}(t)\boldsymbol{\sigma}(\epsilon_t) - b(\epsilon_t)\kappa(t)\boldsymbol{\rho}(\epsilon_t)\big]d\boldsymbol{W}'(t) \\ &\quad - \sqrt{1-|\boldsymbol{\rho}(\epsilon_t)|^2}\, b(\epsilon_t)\kappa(t)d\bar{W}(t) - \gamma(\epsilon_t)\kappa(t)dN(t),\end{aligned} \tag{1}$$

with $X^u(0) > 0$.

We denote by $\mathcal{A}_{t,x,i}$ the set of all admissible controls under the initial conditions $X^u(t) = x$ and $\epsilon(t) = i$, where $t \in [0,T]$, $x > 0$ and $i \in \mathcal{S}$. For every $u \in \mathcal{A}_{t,x,i}$, u is a predictable process and satisfies for every $t \in [0,T]$:

$$E\left[\int_t^T (\pi^m(s))^2 ds\right] < \infty, \text{ for every } m \in \mathcal{K},$$

$$E\left[\int_t^T (\kappa(s))^2 ds\right] < \infty,$$

and

$$0 \le \kappa(s) < \frac{1}{\gamma(\epsilon_s)}, \text{ for every } s \in [t,T].$$

According to equation (2) below, we remark that the last condition above says that the insurer selects her/his liability ratio κ so that bankruptcy does not occur at jumps.

Since all the coefficients of dt, $d\boldsymbol{W}'$, $d\bar{W}$ and dN in the SDE (1) are bounded for every $u \in \mathcal{A}_{t,x,i}$, according to Oksendal and Sulem (2005, Theorem 1.19), there exists a unique strong solution X^u to the SDE (1), and

$$E[|X^u(s)|^2] < \infty \quad \text{for every } s \in [t,T].$$

We define the criterion functional J by

$$J(t,x,i;u) := E_{t,x,i}\big[U(X^u(T))\big],$$

where the utility function U is strictly increasing and concave, and satisfies the linear growth condition

$$\exists C > 0 \text{ such that } U(y) \le C(1+y) \text{ for every } y > 0.$$

The notation $E_{t,x,i}$ means conditional expectation given $X^u(t) = x$ and $\epsilon(t) = i$ under the probability measure $\mathbb{P}$.

We then formulate the optimal investment and liability ratio problem as follows.

Problem 2.1 *Select an admissible control* $u^* = (\boldsymbol{\pi}^*, \kappa^*) \in \mathcal{A}_{t,x,i}$ *that attains the value function* V*, defined by*

$$V(t, x, i) := \sup_{u \in \mathcal{A}_{t,x,i}} J(t, x, i; u).$$

The control u^* *is called an optimal control or an optimal policy.*

Zou and Cadenillas (2014a) studied the special case in which there is only one risky asset (stock) in the financial market, and neither the financial market nor the risk process was affected by the regime of the economy. In other words, our model generalizes the model of Zou and Cadenillas (2014a) by allowing more than one risky asset (stock) in the financial market (we allow $K \geq 1$), and by allowing the regime of the economy to affect both the financial market and the risk process (we allow $S \geq 1$).

3 The Verification Theorem

Let $\psi(t, x, i)$ be a $C^{1,2}$ function for every $i \in \mathcal{S}$. We define the operator $\mathcal{L}_i^u$ by

$$\begin{aligned}\mathcal{L}_i^u \psi &:= \psi_t(t, x, i) + [(\boldsymbol{\mu}_i - r_i \mathbf{1}) \cdot \boldsymbol{\pi}' + r_i + (p_i - a_i)\kappa]\, x \psi_x(t, x, i) \\ &\quad + \frac{1}{2}\Big[|\boldsymbol{\pi} \cdot \boldsymbol{\sigma}_i - b_i \kappa \boldsymbol{\rho}_i|^2 + \big(1 - |\boldsymbol{\rho}_i|^2\big)\, b_i^2 \kappa^2\Big]\, x^2 \psi_{xx}(t, x, i),\end{aligned}$$

where $u := (\boldsymbol{\pi}, \kappa) \in \mathbb{R}^K \times [0, \frac{1}{\gamma_i})$.

Theorem 3.1 *Let* $v(\cdot, \cdot, i) \in C^{1,2}$ *and* $v(t, \cdot, i)$ *be an increasing and concave function for every* $t \in [0, T]$ *and* $i \in \mathcal{S}$*. If* $v(t, x, i)$ *satisfies the Hamilton-Jacobi-Bellman equation*

$$\sup_{u \in \mathbb{R}^K \times [0, \frac{1}{\gamma_i})} \Big\{\mathcal{L}_i^u v(t, x, i) + \lambda[v(t, (1 - \gamma_i \kappa)x, i) - v(t, x, i)]\Big\} = -\sum_{j \in \mathcal{S}} q_{ij} v(t, x, j) \tag{2}$$

and the boundary condition

$$v(T,x,i) = U(x) \tag{3}$$

for every $x>0$, $i\in\mathcal{S}$, *and the control* $u^* = (\boldsymbol{\pi}^*, \kappa^*)$ *defined by*

$$u^* = \arg \sup_{u\in\mathbb{R}^K\times[0,\frac{1}{\gamma_i})} \Big\{ \mathcal{L}_i^u v(t,x,i) + \lambda[v(t,(1-\gamma_i\kappa)x,i) - v(t,x,i)] \Big\}$$

is admissible, then u^* *is optimal control to Problem 2.1 and* $v(t,x,i)$ *is the value function.*

Proof. For every $u\in\mathcal{A}_{t,x,i}$, by applying the Markov-modulated Ito's formula (see, e.g., Sotomayor and Cadenillas (2009)), we obtain, for every $\theta\in[t,T]$,

$$\begin{aligned} v(\theta, X_\theta^u, \epsilon_\theta) = v(t, X_t^u, \epsilon_t) &+ \int_t^\theta \left(\mathcal{L}_{\epsilon_s}^{u(s)} v(s, X_s^u, \epsilon_s) + \sum_{j\in\mathcal{S}} q_{\epsilon_s,j} v(s, X_s^u, j) \right) ds \\ &+ \int_t^\theta X_{s-}^u v_x(s, X_{s-}^u, \epsilon_s) \big(\boldsymbol{\pi}(s)\boldsymbol{\sigma}(s) - b(\epsilon_s)\kappa(s)\boldsymbol{\rho}(s) \big) \cdot d\boldsymbol{W}'(s) \\ &+ \int_t^\theta \Big(v(s, (1-\gamma(\epsilon_s)\kappa_s) X_{s-}^u, \epsilon_s) - v(s, X_{s-}^u, \epsilon_s) \Big) dN_s + m_\theta^v, \end{aligned}$$

where m^v is a square-integrable martingale and $m_0^v = 0$.

For every $u\in\mathcal{A}_{t,x,i}$, X_s^u, π_s^m and κ_s are square integrable for every $m\in\mathcal{K}$ and $s\in[t,T]$. By assumption, v_x is bounded on $[t,T]$. Hence, taking conditional expectation (under $X^u(t) = x$ and $\epsilon_t = i$) for the above Ito integral gives 0.

The function $v(s,(1-\gamma(\epsilon_s)\kappa_s)X_{s-}^u,\epsilon_s) - v(s, X_{s-}^u, \epsilon_s)$ is left continuous and bounded, thus

$$E_{t,x,i}\left[\int_t^\theta \Big(v(s, (1-\gamma(\epsilon_s)\kappa_s) X_{s-}^u, \epsilon_s) - v(s, X_{s-}^u, \epsilon_s) \Big) dM_s \right] = 0,$$

where M, defined as $M_t := N_t - \lambda t$, is the compensated Poisson process of N, and then a true martingale under measure $\mathbb{P}$.

Hence, taking conditional expectation for $v(\theta, X^u_\theta, \epsilon_\theta)$ yields

$$
\begin{aligned}
E_{t,x,i}\big[v(\theta, X^u_\theta, \epsilon_\theta)\big] = v(t,x,i) & \\
+ E_{t,x,i}\Bigg[\int_t^\theta \Big(\sum_{j\in\mathcal{S}} q_{\epsilon_s,j} v(s, X^u_s, j) + \mathcal{L}^{u(s)}_{\epsilon_s} v(s, X^u_s, \epsilon_s) & \\
+ \lambda\big[v(s,(1-\gamma(\epsilon_s)\kappa_s)X^u_s, \epsilon_s) - v(s, X^u_s, \epsilon_s)\big]\Big)ds\Bigg], &
\end{aligned}
$$

which directly implies the HJB equation (2). Then v defined in Theorem 3.1 is the value function to Problem 2.1. Given u^* is admissible, u^* is an optimal control to Problem 2.1 (See, e.g., Fleming and Soner (1993, Chapter III), Oksendal and Sulem (2005, Chapter 3) for analysis). □

4 Construction of Explicit Solutions

In this section, we obtain explicit solutions to Problem 2.1 in a regime switching model. Our strategy is to conjecture that the value function is strictly increasing and strictly concave. Such conjecture will give a candidate for value function and a candidate for optimal control. Next, we will apply the Verification Theorem 3.1 to prove that the candidate for value function is indeed the value function and the candidate for optimal control is indeed the optimal control.

To obtain a candidate for optimal control, we separate the optimization problem in the HJB equation (2) into two sub-optimization problems:

$$
\max_{\boldsymbol{\pi}\in\mathbb{R}^K}\Big[(\boldsymbol{\mu}_i - r_i\mathbf{1})\cdot\boldsymbol{\pi}' x v_x(t,x,i) + \frac{1}{2}|\boldsymbol{\pi}\cdot\boldsymbol{\sigma}_i - b_i\kappa\boldsymbol{\rho}_i|^2 x^2 v_{xx}(t,x,i)\Big]
$$

for investment portfolio $\boldsymbol{\pi}$, and

$$
\begin{aligned}
\max_{\kappa\in[0,\frac{1}{\gamma_i})}\Big[(p_i - a_i)\kappa x v_x(t,x,i) + \frac{1}{2}|\boldsymbol{\pi}\cdot\boldsymbol{\sigma}_i - b_i\kappa\boldsymbol{\rho}_i|^2 x^2 v_{xx}(t,x,i) & \\
+ \frac{1}{2}b_i^2\left(1-|\boldsymbol{\rho}_i|^2\right)\kappa^2 x^2 v_{xx}(t,x,i) + \lambda v(t,(1-\gamma_i\kappa)x,i)\Big] &
\end{aligned}
$$

for liability ratio κ.

Under the conjecture that $v(t,\cdot,i)$ is strictly increasing and strictly concave, we obtain the candidate for $\boldsymbol{\pi}^*$ as

$$\boldsymbol{\pi}^* = -\frac{v_x(t,x,i)}{xv_{xx}(t,x,i)}(\boldsymbol{\mu}_i - r_i\mathbf{1})\boldsymbol{\Sigma}_i^{-1} + b_i\boldsymbol{\rho}_i\boldsymbol{\sigma}_i^{-1}\kappa^*, \tag{4}$$

while the candidate of κ^* is given by

$$\begin{aligned} &xv_{xx}(t,x,i)(1-|\boldsymbol{\rho}_i|^2)b_i^2\kappa^* - \lambda\gamma_i v_x(t,(1-\gamma_i\kappa^*)x,i) \\ &+ v_x(t,x,i)\left(p_i - a_i + b_i(\boldsymbol{\mu}_i - r_i\mathbf{1})(\boldsymbol{\sigma}_i')^{-1}\boldsymbol{\rho}_i'\right) = 0. \end{aligned} \tag{5}$$

We will impose the technical condition

$$p_i - a_i + b_i(\boldsymbol{\mu}_i - r_i\mathbf{1})(\boldsymbol{\sigma}_i')^{-1}\boldsymbol{\rho}_i' > \lambda\gamma_i \qquad \text{for every } i \in \mathcal{S}. \tag{6}$$

We will see below that this inequality guarantees that the equation (5) has a unique solution.

We consider three utility functions

1. $U(x) = \ln(x)$, $x > 0$;
2. $U(x) = \frac{1}{\alpha}x^\alpha$, $x > 0$, where $\alpha < 1$ and $\alpha \neq 0$;
3. $U(x) = -\frac{1}{\alpha}e^{-\alpha x}$, where $\alpha > 0$.

4.1 $U(x) = \ln(x)$, $x > 0$

In this case, we conjecture the solution to the HJB equation (2) is given by

$$v(t,x,i) = \ln(x) + g(t,i),$$

where $g(t,i)$ will be determined below.

We obtain $v_x(t,x,i) = \frac{1}{x}$ and $v_{xx}(t,x,i) = -\frac{1}{x^2}$. Hence, the candidate policy is given by

$$\boldsymbol{\pi}^* = (\boldsymbol{\mu}_i - r_i\mathbf{1}) \cdot \boldsymbol{\Sigma}_i^{-1} + b_i\boldsymbol{\rho}_i\boldsymbol{\sigma}_i^{\mathbf{-1}}\kappa^*, \tag{7}$$

and

$$A_i(\kappa^*)^2 - B_i\,\kappa^* + C_i = 0, \tag{8}$$

where

$$\begin{aligned} A_i &:= (1-|\boldsymbol{\rho}_i|^2)b_i^2\gamma_i, \\ B_i &:= (1-|\boldsymbol{\rho}_i|^2)b_i^2 + \gamma_i\left[p_i - a_i + b_i(\boldsymbol{\mu}_i - r_i\mathbf{1})(\boldsymbol{\sigma}_i')^{-1}\boldsymbol{\rho}_i'\right], \\ C_i &:= p_i - a_i + b_i(\boldsymbol{\mu}_i - r_i\mathbf{1})(\boldsymbol{\sigma}_i')^{-1}\boldsymbol{\rho}_i' - \lambda\gamma_i. \end{aligned} \tag{9}$$

Lemma 4.1 *If the technical condition* (6) *holds, or equivalently* $C_i > 0$ *for every* $i \in \mathcal{S}$*, then there exists a unique solution* $\kappa^* \in [0, \frac{1}{\gamma_i})$ *to the equation* (8).

Proof. Define the function $f_i(x) := A_i x^2 - B_i\, x + C_i$. Then we obtain

$$f_i(0) = C_i > 0, \text{ and } f_i\left(\frac{1}{\gamma_i}\right) = -\lambda\gamma_i < 0,$$

which gives the desired result. □

By substituting candidate strategies $\boldsymbol{\pi}^*$ and κ^*, given by (7) and (8), into the HJB equation (2), we obtain the following system of linear differential equations:

$$g_t(t,i) + \sum_{j\in\mathcal{S}} q_{ij}\, g(t,j) + \Pi_i = 0, \tag{10}$$

where Π_i is defined by

$$\begin{aligned}\Pi_i :&= r_i + \frac{1}{2}(\boldsymbol{\mu}_i - r_i\mathbf{1})\boldsymbol{\Sigma}_i^{-1}(\boldsymbol{\mu}_i - r_i\mathbf{1})' + \lambda\ln(1-\gamma_i\kappa^*)\\ &\quad + \big(p_i - a_i + b_i(\boldsymbol{\mu}_i - r_i\mathbf{1})(\boldsymbol{\sigma}_i')^{-1}\boldsymbol{\rho}_i'\big)\kappa^* - \frac{1}{2}b_i^2(1-|\boldsymbol{\rho}|^2)(\kappa^*)^2.\end{aligned}$$

In addition, g also satisfies the boundary condition

$$g(T,i) = 0 \qquad \text{for every } i \in \mathcal{S}.$$

Notice that the linear ODE system (10) has a unique solution $g(t,i)$ and the candidate for optimal control, given by (7) and (8), is square integrable. Hence, the candidate for optimal control is indeed the optimal control and $v(t,x,i) = \ln(x) + g(t,i)$ is the value function to Problem 2.1.

Theorem 4.1 *Consider the case* $U(x) = \ln(x)$*. Then,* $u^* = (\boldsymbol{\pi}^*, \kappa^*)$*, where* $\kappa^*(t)$ *is the unique solution in* $\left[0, \frac{1}{\gamma(\epsilon(t))}\right)$ *to the equation*

$$A_{\epsilon_t}(\kappa^*(t))^2 - B_{\epsilon_t}\,\kappa^*(t) + C_{\epsilon_t} = 0,$$

and

$$\boldsymbol{\pi}^*(t) = (\boldsymbol{\mu}(t) - r(\epsilon_t)\mathbf{1})\cdot\boldsymbol{\Sigma}^{-1}(\epsilon_t) + b(\epsilon_t)\boldsymbol{\rho}(t)\boldsymbol{\sigma}^{-1}(\epsilon_t)\kappa^*(t),$$

is optimal control to Problem 2.1.

4.2 $U(x) = \frac{1}{\alpha}x^{\alpha}$, $x > 0$, where $\alpha < 1$ and $\alpha \neq 0$

In this case, the utility function is of constant relative risk aversion (CRRA) type and the relative risk aversion coefficient is $1 - \alpha$.

The solution to the HJB (2) is given by

$$v(t, x, i) = \frac{1}{\alpha} x^{\alpha} \cdot \hat{g}(t, i),$$

where $\hat{g}(t, i) > 0$ for every $i \in \mathcal{S}$ will be determined below.

From equations (4)-(5), we obtain the candidate for optimal control κ^* is a solution to the equation

$$(\alpha-1)(1-|\boldsymbol{\rho}_i|^2)b_i^2\kappa^* - \lambda\gamma_i(1-\gamma_i\kappa^*)^{\alpha-1} + p_i - a_i + b_i(\boldsymbol{\mu}_i - r_i\mathbf{1})(\boldsymbol{\sigma}_i')^{-1}\boldsymbol{\rho}_i' = 0, \quad (11)$$

and

$$\boldsymbol{\pi}^* = \frac{1}{1-\alpha}(\boldsymbol{\mu}_i - r_i\mathbf{1}) \cdot \boldsymbol{\Sigma}_i^{-1} + b_i\boldsymbol{\rho}_i\boldsymbol{\sigma}_i^{-1}\kappa^*. \quad (12)$$

Lemma 4.2 *If the condition* (6) *holds, then there exists a unique solution in* $[0, \frac{1}{\gamma_i})$ *to the equation* (11).

Proof. Let $\phi_i := 1 - \gamma_i\kappa^*$ and define $\hat{B}_i$ and $\hat{C}_i$ by

$$\begin{aligned} \hat{B}_i &:= \frac{(\alpha-1)(1-|\boldsymbol{\rho}_i|^2)b_i^2}{\lambda\gamma_i^2}, \\ \hat{C}_i &:= -\frac{1}{\lambda\gamma_i}\left[p_i - a_i + b_i(\boldsymbol{\mu}_i - r_i\mathbf{1})(\boldsymbol{\sigma}_i')^{-1}\boldsymbol{\rho}_i'\right] - \hat{B}_i. \end{aligned} \quad (13)$$

Then to show there exists a unique solution in $[0, \frac{1}{\gamma_i})$ to the equation (11), we only need to prove that the following equation has a unique solution in $(0, 1]$

$$\phi_i^{\alpha-1} + \hat{B}_i\phi_i + C_i = 0.$$

Consider the function $\hat{h}_i(x) := x^{\alpha-1} + \hat{B}_i x + C_i$. At the two end points, we have

$$\begin{aligned} \hat{h}_i(0) &= \lim_{x\to 0^+} \hat{h}_i(x) = +\infty, \\ \hat{h}_i(1) &= 1 - \frac{1}{\lambda\gamma_i}\left[p_i - a_i + b_i(\boldsymbol{\mu}_i - r_i\mathbf{1})(\boldsymbol{\sigma}_i')^{-1}\boldsymbol{\rho}_i'\right] < 0, \end{aligned}$$

where the above inequality comes from the condition (6).

Furthermore, we have $\hat{h}_i'(x) = (\alpha-1)x^{\alpha-2}+B_i < 0$ and $\hat{h}_i(x)$ is continuous in $(0,1)$, which together give the desired result. $\square$

By plugging the candidate for optimal control into the HJB equation (2), we obtain

$$\hat{g}_t(t,i) + \sum_{j\in\mathcal{S}} q_{ij}\hat{g}(t,j) + \alpha\,\hat{\Pi}_i\,\hat{g}(t,i) = 0, \tag{14}$$

where $\hat{\Pi}$ is defined by

$$\begin{aligned}\hat{\Pi}_i := {} & r_i + \frac{1}{2(1-\alpha)}(\boldsymbol{\mu}_i - r_i\mathbf{1})\boldsymbol{\Sigma}_i^{-\mathbf{1}}(\boldsymbol{\mu}_i - r_i\mathbf{1})' + \lambda\left[(1-\gamma_i\kappa^*)^\alpha - 1\right] \\ & + \left[p_i - a_i + b_i(\boldsymbol{\mu}_i - r_i\mathbf{1})(\boldsymbol{\sigma}_i')^{-1}\boldsymbol{\rho}_i'\right]\kappa^* - \frac{1}{2}(1-\alpha)b_i^2(1-|\boldsymbol{\rho}|^2)(\kappa^*)^2.\end{aligned}$$

The boundary condition for $\hat{g}$ is given by

$$\hat{g}(T,i) = 1 \qquad \text{for every } i \in \mathcal{S}.$$

We remark that the above linear ODE system has a unique solution. Furthermore, to verify our conjecture that $v(t,\cdot,i)$ is strictly increasing and strictly concave, we need to show that $\hat{g}(t,i)$ is strictly positive for every $i \in \mathcal{S}$, which is given by Lemma 4.3.

Lemma 4.3 *The function $\hat{g}(t,i)$, solution to the equation* (14)*, is strictly positive.*

Proof. Using Ito's formula for Markov-modulated process, we obtain

$$\hat{g}(T,\epsilon_T) = \hat{g}(t,\epsilon_t) + \int_t^T \hat{g}_t(s,\epsilon_s)ds + \int_t^T \sum_{j\in\mathcal{S}} q_{\epsilon_s,j}\hat{g}(s,j)ds + m_T^{\hat{g}},$$

where $m^{\hat{g}}$ is a square integrable martingale with $m_0^{\hat{g}} = 0$.

Taking conditional expectation and using the equation (14), we get

$$E_{t,x,i}[\hat{g}(T,\epsilon_T)] = \hat{g}(t,i) - E_{t,x,i}\left[\int_t^T \alpha\hat{\Pi}_{\epsilon(s)}\hat{g}(s,\epsilon_s)ds\right],$$

which is equivalent to (recall the boundary condition $\hat{g}(T,i) = 1$)

$$\hat{g}(t,i) = 1 + E_{t,x,i}\left[\int_t^T \alpha\hat{\Pi}_{\epsilon(s)}\hat{g}(s,\epsilon_s)ds\right].$$

We find the unique solution to be given by

$$\hat{g}(t,i) = E_{t,x,i}\left[\exp\left\{\int_t^T \alpha\hat{\Pi}_{\epsilon(s)}ds\right\}\right].$$

Hence, the positiveness of $\hat{g}(t,i)$ follows. $\square$

From the construction of $\hat{g}(t,i)$ and Lemma 4.3, $v(t,x,i) = \frac{1}{\alpha}x^\alpha \cdot \hat{g}(t,i)$ is the candidate for the value function to Problem 2.1. According to Lemma 4.2, $u^* = (\boldsymbol{\pi}^*, \kappa^*)$ is admissible, where $\boldsymbol{\pi}^*$ and κ^* are given by (12) and (11). Hence Theorem 4.2 follows accordingly.

Theorem 4.2 *Consider the case $U(x) = \frac{1}{\alpha}x^\alpha$, where $\alpha < 1$ and $\alpha \neq 0$. Then $u^* = (\boldsymbol{\pi}^*, \kappa^*)$ is optimal control to Problem 2.1, where $\kappa^*(t)$ is the unique solution to the equation*

$$\begin{aligned}&(\alpha-1)(1-|\boldsymbol{\rho}(\epsilon_t)|^2)b^2(\epsilon_t)\kappa^*(t) - \lambda\gamma(\epsilon_t)(1-\gamma(\epsilon_t)\kappa^*(t))^{\alpha-1}\\ &+ p(\epsilon_t) - a(\epsilon_t) + b(\epsilon_t)(\boldsymbol{\mu}(\epsilon_t) - r(\epsilon_t)\mathbf{1})(\boldsymbol{\sigma}')^{-1}(\epsilon_t)\boldsymbol{\rho}_i' = 0,\end{aligned}$$

and

$$\boldsymbol{\pi}^*(t) = \frac{1}{1-\alpha}(\boldsymbol{\mu}(\epsilon_t) - r(\epsilon_t)\mathbf{1})\cdot\boldsymbol{\Sigma}^{-1}(\epsilon_t) + b(\epsilon_t)\boldsymbol{\rho}_i\boldsymbol{\sigma}^{-1}(\epsilon_t)\kappa^*(t).$$

4.3 $U(x) = -\frac{1}{\alpha}e^{-\alpha x}$, where $\alpha > 0$

In this subsection, we consider the exponential utility function, which belongs to the class of constant absolute risk aversion (CARA) utility functions. We conjecture the solution to the HJB (2) is of the form

$$v(t,x,i) = -\frac{1}{\alpha}\exp\left\{-\alpha e^{r_i(T-t)}x + \tilde{g}(t,i)\right\},$$

where $\tilde{g}(t,i)$ will be determined below.

For the above solution, we calculate that

$$\begin{aligned}v_t(t,x,i) &= \left[\alpha\, r_i x e^{r_i(T-t)} + \tilde{g}_t(t,i)\right] v(t,x,i),\\ v_x(t,x,i) &= -\alpha\, e^{r_i(T-t)}v(t,x,i),\\ v_{xx}(t,x,i) &= \alpha^2\, e^{2r_i(T-t)}v(t,x,i).\end{aligned}$$

Hence, we obtain the candidate for $\boldsymbol{\pi}^*$

$$\boldsymbol{\pi}^* = \frac{1}{\alpha x e^{r_i(T-t)}}(\boldsymbol{\mu}_i - r_i\mathbf{1})\cdot\boldsymbol{\Sigma}_i^{-1} + b_i\boldsymbol{\rho}_i\boldsymbol{\sigma}_i^{-1}\kappa^*. \tag{15}$$

Apparently, in this case, it is more convenient to use the actual amount instead of the proportion as the control of investment, see, e.g., Browne (1995), Wang et al. (2007), and Yang and Zhang (2005). We then define $\theta^m(t)$ as the amount of money invested in the m-th risky asset and $L(t)$ as the total liabilities at time t. Denote $\boldsymbol{\theta} := (\theta^1, \theta^2, \ldots, \theta^K)$. By definition, we have

$$\boldsymbol{\theta}(t) = \boldsymbol{\pi}(t)X^u(t) \quad \text{and} \quad L(t) = \kappa(t)X^u(t).$$

Thus, by (15) and (5), we obtain the candidate for $\boldsymbol{\theta}^*$

$$\boldsymbol{\theta}^* = \frac{1}{\alpha e^{r_i(T-t)}}(\boldsymbol{\mu}_i - r_i\mathbf{1}) \cdot \boldsymbol{\Sigma}_i^{-1} + b_i\boldsymbol{\rho}_i\boldsymbol{\sigma}_i^{-1}L^*, \tag{16}$$

and the candidate for L^*, which satisfies

$$\lambda\gamma_i e^{\tilde{A}_i L^*} + \tilde{B}_i L^* - \tilde{C}_i = 0, \tag{17}$$

where

$$\begin{aligned}
\tilde{A}_i &:= \alpha\gamma_i e^{r_i(T-t)}, \\
\tilde{B}_i &:= \alpha e^{r_i(T-t)}(1 - |\boldsymbol{\rho}_i|^2)b_i^2, \\
\tilde{C}_i &:= p_i - a_i + b_i(\boldsymbol{\mu}_i - r_i\mathbf{1})(\boldsymbol{\sigma}_i')^{-1}\boldsymbol{\rho}_i'.
\end{aligned}$$

Lemma 4.4 *If the condition* (6) *holds, then there exists a unique solution to the equation* (17).

Proof. Define the function $\tilde{h}_i(x) := \lambda\gamma_i e^{\tilde{A}_i x} + \tilde{B}_i x - \tilde{C}_i$. Then we get

$$\tilde{h}_i'(x) = \lambda\gamma_i\tilde{A}_i e^{\tilde{A}_i x} + \tilde{B}_i > 0,$$

since $\lambda > 0$, $\gamma_i > 0$, $\tilde{A}_i > 0$ and $\tilde{B}_i > 0$ for every $i \in \mathcal{S}$. If the condition (6) is satisfied, we obtain $\tilde{h}_i(0) = \lambda\gamma_i - \tilde{C}_i < 0$. Since both $\tilde{A}_i$ and $\tilde{B}_i$ are positive, $\tilde{h}_i(x)$ must be positive when x is large enough. Therefore, the desired conclusion is obtained. □

Next, we rewrite the HJB equation (2) as follows

$$\tilde{g}_t(t,i) + \sum_{j\in\mathcal{S}} q_{ij}\exp\left\{-\alpha x e^{(r_j - r_i)(T-t)} + \tilde{g}(t,j) - \tilde{g}(t,i)\right\} + \tilde{\Pi}_i = 0, \tag{18}$$

where

$$\begin{aligned}\tilde{\Pi}_i := &\lambda \left[e^{\alpha\gamma_i e^{r_i(T-t)} L^*} - 1 \right] + \frac{1}{2}\alpha^2 e^{2r_i(T-t)}(1-|\boldsymbol{\rho}_i|^2) b_i^2 (L^*)^2 \\ &- \alpha e^{r_i(T-t)} \left[p_i - a_i + b_i(\boldsymbol{\mu}_i - r_i\mathbf{1})(\boldsymbol{\sigma}'_i)^{-1}\boldsymbol{\rho}'_i \right] L^* \\ &- \frac{1}{2}(\boldsymbol{\mu}_i - r_i\mathbf{1})\boldsymbol{\Sigma}_i^{-1}(\boldsymbol{\mu}_i - r_i\mathbf{1})'.\end{aligned}$$

Let $\tilde{q}_{ij} := q_{ij}\exp\left\{-\alpha x e^{(r_j-r_i)(T-t)}\right\}$ and $\Phi(t,i) := \exp\{\tilde{g}(t,i)\}$. Then equation (18) becomes

$$\Phi_t(t,i) + \sum_{j\in\mathcal{S}} \tilde{q}_{ij}\Phi(t,j) + \tilde{\Pi}_i\Phi(t,i) = 0,$$

which, similar to the system (14), bears a unique solution. Hence, there exists a unique solution $\tilde{g}(t,i)$ to the system (18).

Since the candidate for value function is well defined even when $x \le 0$, we remove the constraint $\kappa(t) < \frac{1}{\gamma_{\epsilon_t}}$ from the conditions of the admissible set $\mathcal{A}_{t,x,i}$, which is used in Subsections 4.1 and 4.2 to guarantee $(1-\kappa(t)\gamma_{\epsilon_t}) > 0$ (so that $\ln(1-\kappa(t)\gamma_{\epsilon_t})$ and $(1-\kappa(t)\gamma_{\epsilon_t})^{\alpha-1}$, where $\alpha < 1$, are well defined).

By Lemma 4.4, L^*, given by (17), is finite for every $i \in \mathcal{S}$, which implies L^* is square integrable on $[t,T]$. Hence, by (15), $\boldsymbol{\theta}^*$ is also square integrable on $[t,T]$. By Oksendal and Sulem (2005, Theorem 1.19), there exists a unique wealth process X^{u^*} such that $E[|X^{u^*}(s)|^2] < \infty$ for all $s \in [t,T]$. In consequence, $u^* := \frac{1}{X^{u^*}}(\boldsymbol{\theta}^*, L^*)$ is admissible and the theorem below follows.

Theorem 4.3 *Consider the case in which the utility function is $U(x) = -\frac{1}{\alpha}e^{-\alpha x}$, where $\alpha > 0$. Then, $u^*(t) := \frac{1}{X^{u^*}(t)}(\boldsymbol{\theta}^*(t), L^*(t))$ is optimal control to Problem 2.1, where $L^*(t)$ is the unique solution to the equation*

$$\lambda\gamma_{\epsilon_t}\exp\{\tilde{A}_{\epsilon_t}L^*(t)\} + \tilde{B}_{\epsilon_t}L^*(t) - \tilde{C}_{\epsilon_t} = 0,$$

and $\boldsymbol{\theta}^*$ is given by

$$\boldsymbol{\theta}^*(t) = \frac{1}{\alpha e^{r(\epsilon_t)(T-t)}}(\boldsymbol{\mu}(\epsilon_t) - r(\epsilon_t)\mathbf{1}) \cdot \boldsymbol{\Sigma}^{-1}(\epsilon_t) + b(\epsilon_t)\boldsymbol{\rho}(\epsilon_t)\boldsymbol{\sigma}^{-1}(\epsilon_t)L^*(t).$$

Here X^{u^*} is the unique strong solution to the SDE (1) under u^*.

5 Economic Analysis

In this section, we study the impact of the insurer's risk attitude, the negative correlation ρ, and the regime of the economy, on optimal policy. To this purpose, we assume there are two regimes in the economy. Regime 1 represents a bull market, in which the economy is booming. Regime 2 represents a bear market, meaning the economy is in recession. We take $K = 1$, that means there is only one risky asset in the financial market, and we denote the return and volatility of this risky asset by μ_i and σ_i, respectively. For comparative analysis, we consider HARA utility functions, namely, $U(x) = \frac{1}{\alpha}x^{\alpha}$, where $\alpha < 1$ ($\alpha = 0$ is associated with the case of logarithmic utility function $U(x) = \ln(x)$). Insurers are high risk-averse when $\alpha < 0$, moderate risk-averse when $\alpha = 0$, and low risk-averse when $0 < \alpha < 1$.

We assume $\mu_i > r_i > 0$ and $p_i > a_i > 0$, $i = 1, 2$ (Remark 2.2). French et al. (1987) find that capital returns are higher in a bull market, hence we assume $\mu_1 > \mu_2$ and $r_1 > r_2$. Hamilton and Lin (1996) show that the stock volatility is greater when the economy is in recession, which implies $\sigma_1 < \sigma_2$. Furthermore, we assume $\frac{\mu_1 - r_1}{\sigma_1^2} > \frac{\mu_2 - r_2}{\sigma_2^2}$, as supported by French et al. (1987). Motivated by non-traditional insurance market like CDS, we assume that the risk process (claims) is negatively correlated with the stock returns and interest rate, see, e.g., Haley (1993), Norden and Weber (2007). This conclusion leads to the assumption that $a_2 > a_1$, $b_2 > b_1$, $\gamma_2 > \gamma_1$, and $\rho_2 < \rho_1$. When the economy is in recession, the insurance companies charge a higher premium, hence $p_2 > p_1$. We also notice that the coefficients we choose should satisfy the technical condition (6). Based on the above argument, we choose the parameters as in Table 1.

Regime	μ	r	σ	a	b	γ	p	ρ	λ
1 (bull market)	0.1	0.03	0.15	0.04	0.05	0.2	0.1	-0.3	0.01
2 (bear market)	0.05	0.01	0.25	0.08	0.1	0.5	0.2	-0.5	

Table 1: Market parameters

From Theorem 4.1, we calculate the optimal policy for moderate risk-averse insurers (that is, $\alpha = 0$). For both high risk-averse ($\alpha < 0$) and low risk-averse ($0 < \alpha < 1$) insurers, we obtain the corresponding optimal policy through Theorem 4.2. The results are listed in Table 2.

According to the optimal policy obtained in Table 2, we observe that both

α	Regime	π^* (Investment)	κ^* (Liability Ratio)
-5	1	0.3395	1.7900
	2	-0.0407	0.7370
-2	1	0.7347	3.0233
	2	-0.0308	1.2207
-1	1	1.1740	3.8155
	2	0.0148	1.5262
-0.01	1	2.6048	4.7550
	2	0.2548	1.8945
0	1	2.6348	4.7628
	2	0.2605	1.8977
0.01	1	2.6655	4.7705
	2	0.2663	1.9009
0.1	1	2.9733	4.8350
	2	0.3256	1.9275
0.2	1	3.3994	4.8953
	2	0.4094	1.9529
0.5	1	5.7231	4.9911
	2	0.8809	1.9954
0.7	1	9.8704	4.9999
	2	1.7333	1.9999

Table 2: Impact of α on optimal policies

the optimal investment proportion in the risky asset π^* and the optimal liability ratio κ^* are increasing functions of the risk aversion parameter α. Hence less risk-averse insurers (that is, insurers with large α) invest proportionally more in the risky asset and choose a higher liability ratio.

As pointed out in Stein (2012, Chapter 6), a major mistake that contributed significantly to AIG's sudden collapse was the negligence of the negative correlation between the risk and the capital returns (equivalently, AIG assumed $\rho_i \equiv 0$ instead of $\rho_i < 0$). Hence in the next analysis, we calculate the optimal policy for different values of ρ_i. We still keep all the other parameters unchanged as in Table 1, but consider three cases: (1) $\rho_1 = -0.6,\ \rho_2 = -0.8$; (2) $\rho_1 = -0.3,\ \rho_2 = -0.5$; (3) $\rho_1 = -0.1,\ \rho_2 = -0.2$.

Based on the results in Table 3, we observe that the optimal proportion invested in the risky asset π^* is an increasing function of ρ. However, the relationship between κ^* and ρ is more complicated, some show convexity

ρ	α	Regime	π^* (Investment)	κ^* (Liability Ratio)
	-1	1	0.7985	3.7853
		2	-0.1740	1.5438
Case (1)	0	1	2.1632	4.7397
		2	0.0319	1.9004
	0.5	1	5.2245	4.9887
		2	0.6415	1.9953
	-1	1	1.1740	3.8155
		2	0.0148	1.5262
Case (2)	0	1	2.6348	4.7628
		2	0.2605	1.8977
	0.5	1	5.7231	4.9911
		2	0.8809	1.9954
	-1	1	1.4268	3.8612
		2	0.1982	1.5221
Case (3)	0	1	2.9517	4.7818
		2	0.4881	1.8986
	0.5	1	6.0558	4.9925
		2	1.1203	1.9957

Table 3: Impact of ρ on optimal policies

while other show monotonicity (a similar result can be found in Zou and Cadenillas (2014a) for the case of only one regime).

Furthermore, the dependency of the optimal policy on the regime of the economy is evident. We notice both in Table 2 and Table 3 that $\pi_1^* > \pi_2^*$ and $\kappa_1^* > \kappa_2^*$ for all insurers (all α). This result shows that all insurers take more risk in a bull market by spending a greater proportion on the risky asset and selecting a higher liability ratio.

6 Conclusion

The 2007-2009 financial crisis brought new challenges on risk management to all market participants. AIG followed bad risk management strategies, and as a consequence, almost went bankrupt in 2008. There were two major contributors to AIG's sudden collapse. First, AIG did not pay full attention to the business cycles (regime switching) in the U.S. housing market, which directly caused a significant underestimation of the risk of CDS policies. Second, AIG ignored the negative correlation between its liabilities and the capital gains in the financial market.

To address these two problems in the AIG case, we set up a regime switching model from an insurer's perspective and assume not only the financial market but also the insurer's risk process depend on the regime of the economy. An insurer makes investment decisions in a financial market which consists of a riskless asset and K risky assets, and faces an external risk that is negatively correlated with the capital returns of the risky assets. The insurer wants to maximize her/his expected utility of terminal wealth by selecting simultaneously the optimal investment proportion in the risky assets and the optimal liability ratio. We obtain explicit solutions of optimal investment and liability ratio policies when the insurer's utility is given by logarithmic, power, and exponential utility functions.

Through an economic analysis, we find that the optimal policy depends strongly on the regime of the economy. To be more specific, all insurers should invest a greater proportion on the risky assets and select a higher liability ratio when the market is in bull regime. We also observe that the optimal proportion invested in the risky asset is increasing with respect to both α and ρ. Furthermore, the optimal liability ratio also increases when α rises, but its relation with ρ is not always monotone.

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References

Atkinson, T., Luttrell, D., and Rosenblum, H., 2013. How bad was it? The costs and consequenes of the 2007-09 financial crisis. Federal Reserve Bank of Dallas Staff Papers 20 (July).

Bauerle, N., and Rieder, U., 2004. Portfolio optimization with Markov-modulated stock prices and interest rates. IEEE Transaction on Automatic Control 49(3):,442-447.

Browne, S., 1995. Optimal investment policies for a firm with a random risk process: exponential utility and minimizing the probability of ruin. Mathematics of Operations Research 20, 937-958.

Fleming, W., and Soner, H., 1993. Controlled Markov Processes and Viscosity Solutions. Springer.

French, K., Schwert, G., and Stambaugh, R., 1987. Expected stock returns and volatility. Journal of Financial Economics 19, 3-29.

Haley, J., 1993. A cointegration analysis of the relationship between underwriting margins and interest rates: 1930-1989. Journal of Risk and Insurance 60(3), 480-493.

Hamilton, J., and Lin, G., 1996. Stock market volatility and the business cycle. Journal of Applied Econometrics 11, 573-593.

Harrington, S., 2009. The financial crisis, systemic risk and the future of insurance regulation. The Journal of Risk and Insurance 76, 785-819.

Hojgaard, B., and Taksar, M., 1998. Optimal proportional reinsurance policies for diffusion models. Scandinavian Actuarial Journal 2, 166-180.

Kaluszka, M., 2001. Optimal reinsurance under mean-variance premium principles. Insurance: Mathematics and Economics 28, 61-67.

Karatzas, I., Lehoczky, J., Sethi, S., and Shreve, S., 1986. Explicit solution of a general consumption/investment problem. Mathematics of Opeations Research 11(2), 261-294.

Karatzas, I., and Shreve, S., 1998. Methods of Mathematical Finance, Springer-Verlag.

Liu, J., Yiu, K., Siu, T.K., and Ching, W., 2013. Optimal investment-reinsurance with dynamic risk constraint and regime switching. Scandinavian Actuarial Journal 4, 263-285.

Merton, R., 1969. Lifetime portfolio selection under uncertainty: the continuous time case. Review of Economics and Statistics 51, 247-257.

Moore, K., and Young, V., 2006. Optimal insurance in a continuous-time model. Insurance: Mathematics and Economics 39, 47-69.

Norden, L., and Weber, M., 2007. The co-movement of credit default swap, bond and stock markets: an empirical analysis. European Financial Management 15(3), 529-562.

Oksendal, B., and Sulem, A., 2005. Applied Stochastic Control of Jump Diffusions, Springer.

Schmidli, H., 2001. Optimal proportional reinsurance policies in a dynamic setting. Scandinavian Actuarial Journal 1, 55-68.

Sethi, S., 1997. Optimal consumption and investment with bankruptcy. Springer.

Sjostrom, W., 2009. The AIG Bailout. Washington and Lee Law Review 66, 943-991.

Sotomayor, L., and Cadenillas, A., 2009. Explicit solutions of consumption investment problems in financial market with regime switching. Mathematical Fiance 19(2), 251-279.

Stein, J., 2012. Stochastic Optimal Control and the U.S. Financial Debt Crisis, Springer.

Taksar, M., 2000. Optimal risk and dividend distribution control models for an insurance company. Mathematical Methods of Operations Research 51, 1-42.

Wang, Z., Xia, J., and Zhang, L., 2007. Optimal investment for an insurer: The martingale approach. Insurance: Mathematics and Economics 40, 322-334.

Yang, H., and Zhang, L., 2005. Optimal investment for insurer with jump-diffusion risk process. Insurance: Mathematics and Economics 37, 615-634.

Zhou, X., and Yin, G., 2004. Markowitz's Mean-Variance Portfolio Selection with Regime Switching: A Continuous-Time Model. SIAM Journal on Control and Optimization 42, 1466-1482.

Zhuo, J., Yin, G., and Wu, F., 2013. Optimal reinsurance strategies in regime-switching jump diffusion models: Stochastic differential game formulation and numerical methods. Insurance: Mathematics and Economics 53(3), 733-746.

Zou, B., 2014. Stochastic Control in Optimal Insurance and Investment with Regime Switching. PhD Dissertation. Department of Mathematical and Statistical Sciences, University of Alberta, Canada.

Zou, B., and Cadenillas, A., 2014a. Optimal investment and risk control problem for an insurer: expected utility maximization. Insurance: Mathematics and Economics 58, 57-67.

Zou, B., and Cadenillas, A., 2014b. Explicit solutions of optimal consumption, investment and insurance problem with regime switching. Insurance: Mathematics and Economics 58, 159-167.