Embedding the Natural Hedging of Mortality/Longevity Risks into Product Design

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ABSTRACT

How to manage mortality/longevity risks is essential to the long-term solvency of life insurance companies. The literatures proposed to hedge the products subject to the longevity risk (such as annuities) by using the products subject to the mortality risk (e.g., whole life insurance) sold by an insurer. Such natural hedging is intuitive but may be difficult to implement due to the rigid sales market and incentive issues. We propose to embed natural hedging into product design so that the hedging may occur within a product. The key is to offset the impact of mortality on the timing of death that in turn determines the present value of the death benefit by cleverly choose the growth rate of the death benefit. how much to pay while δ would reflect the time value of payment. We provide theoretical derivations, graphical illustrations, and numerical analyses to illustrate the idea of embedding natural hedging into product design.

Keywords: longevity risk; mortality risk; natural hedging

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1. Introduction

The risk emerging from the changes of mortality rate is a critical problem to the insurance providers. When a quick spread of deadly contagious disease causes a great deal of casualty the mortality rate would increase and the mortality risk rises up. In general timing, the mortality rate is decreasing gradually due to progressive medical technology nowadays. That makes longevity risk rise up. Both mortality risk and longevity risk are mortality rate risks caused by the dynamic changes of mortality rate.

Whether it is mortality risk or longevity risk, the mitigation of mortality rate risk has a lot of discussion in literatures. The longevity risk catches more attention owing to its persistent trend. As the exceeding living becomes a normal case to everyone in the world, it is evolving into a systemic risk. On the other hand, the dynamic pattern of mortality risk is focusing on a sudden mortality rate shock. Those risks are not only a risk to the product providers but also a costly problem to economic society. The technique of risk mitigation in both risks is an important issue to risk management.

Considering how to mitigate mortality rate risk, many studies start with the longevity risk. Transferring longevity risk externally with the financial vehicles of capital market is the earliest suggestion in literatures to solve the problem. Blake and Burrows (2001) propose a solution accordingly through capital market by issuing survivor bonds to mitigate the longevity risk exposed to living benefit providers. Other prior studies also provide mitigation solutions through capital market, including survivor bonds (Denuit, Devolder, and Goderniaux, 2007), survivor swaps (e.g. Dowd and Blake, 2006; Dowd, Blake, Cairns, and Dawson, 2006), mortality swaps (Lin and Cox, 2007), mortality securitization (e.g. Dowd, 2003; Lin and Cox, 2005; Cairns, Blake, and Dowd, 2006a; Blake, Cairns, and Dowd, 2006; Cox, Lin, and Wang, 2006; Blake, Cairns, and Dowd, 2006; Blake, Cairns, Dowd, and MacMinn, 2006). These studies suggest mitigating longevity risk by transferring the risk to the investors of capital market. Transferring the risk externally may be a possible way but may not diminish the primary risk rooted in the policy. These methods are also involved with uncertainty in market environment and transaction cost to the providers.

Natural hedging strategy is an alternative. Instead of transferring risk outside the initiative risk bearing company, some studies take steps to alleviate risk by way of inner mitigation within the company. Natural hedging strategy makes risk mitigation executed in the insurance company internally. A whole life insurance product is exposed to mortality risk and an annuity product is exposed to longevity risk. Companies can take advantage to diminish the threat of mortality rate risk through selling life insurance products and annuity products simultaneously.

The main stream of the natural hedging strategy in studies is to optimize the proportion of a product portfolio that helps maximize the hedging effect. Cox and Lin (2007) create a product portfolio with a life insurance product and an annuity product that shows the existence of natural hedging effect between life insurance product and annuity product. Wang et al. (2010) soon propose an immunization model to achieve an optimal life insurance to annuity ratio in mitigating longevity risk. They evaluate the hedging effect to the reserves of a product portfolio and utilize matching of durations and convexities to generate the optimal ratio of life insurance to annuity product by the impact of mortality rate changes. In most recent studies Tsai and Chung (2013) and, Lin and Tsai (2013) adopt the duration/convexity matching to the prices of life insurance and annuity products and extend the application in determining the weights of two or three products in a product portfolio that can obtain the maximum effect of mitigation.

Besides mortality duration/convexity technique in determining optimal natural hedging strategy, Tsai, Wang, and Tzeng (2010) also take the variance of a product portfolio into consideration and generate a narrower quantile of loss distribution compared to immunization model (Wang, Huang, and Hong, 2013). Furthermore, Wang, Huang, and Hong (2013) employ a portfolio of zero coupon bonds, life insurance policies, and annuity policies to construct a feasible objective function that makes possible to catch the mispricing effect and the variance effect of the entire company portfolio. They also address a natural hedging strategy practically.

We find a way of natural hedging strategy within a policy that we can immunize/mitigate the mortality rate risk through delicate product design. We are able to optimize the risk mitigation in a simply death benefit protection product and have not to build up a weighted portfolio of life insurance and annuity. Every insurance claim contains at least two key factors of “when to pay” and “how much to pay” at the events written in policy but normally people overlook “how much to pay” can be a variable related to risk exposure. We may set the death benefit as a function of face amount and vary along with time in the content of product design.

We attempt to utilize both the timing of benefit and the amount of benefit to be variables in determining the risk of the product. If we design the product in which the risk resulting from “when to pay” is more/less than the expected and that from “how much to pay” is less/more than the expected accordingly, we may immunize the risk within the product. The expected value is determined by the present value of the policy cash flows and the cash flows are decided by the future lifetime distribution based on mortality assumption.

The time value is the bridge between “when to pay” and “how much to pay”. The risk rooting in “when to pay” controlled by the factor, the force of interest rate, can be represented the time value of death benefit. If the death benefit is paid earlier than expected timing the company faces losses of the time value. In this case, we may envisage that the amount of benefit is paid less in response to the loss of time value. Thus, we introduce a new factor, the force of amount, in determining the amount of death benefit and it is also engaged to the risk rooting in “how much to pay”. We thus are able to create the product that when the benefit is paid earlier/later than expected timing its amount of benefit is paid less/more accordingly.

We implement our strategies by assessment of risk in product design as illustrated products in this article, in which we present the optimal strategy of perfect mitigation and the secondary strategy in diminishing risk. Our strategies through risk immunized product facilitate the marketing and management and also make mortality risk mitigation feasible to carry on. Following our strategies in product design, the insurance product has a lot more possibility in engaging to financial instrument without considering the mortality rate risk. Our finding can also provide a further research or re-design of previous studies that ignore the existence of a mortality rate risk in an insurance product valuation or risk management consideration. As far as we know, there are no such approaches in the field of the natural hedging literature yet.

The remainder of this article is organized as follows. In the section “Theoretical Development”, we deduce and elaborate our theoretical development for a natural hedging strategy using the Laplace transform that we generate a specified life insurance. In the next section “Numerical Analysis” we use an existing model with U.S. mortality experience and demonstrate how to implement our proposed strategies to assess the natural hedging effect with numerical examples in the section. We extend our analysis to the cases of annuity product in the next section, and then we conclude in the last section.

2. Theoretical Development

2.1 Idea Scratching

We first analyze a traditional whole life insurance product in the framework of Laplace transform[2]. The net single premium for the whole life insurance with 1- unit face amount expressed by the notations of Bowers et al. (1997) can also be expressed in Laplace transform format as:

$$\int_0^\infty e^{-\delta s}\, b(s)\ {}_s p_x\ \mu_{x+s}\, ds\ = \mathcal{L}\left\{b(s)\cdot f(s)\right\} = \phi\left(\delta\right), \tag{1}$$

where δ is the force of interest, $b(s)$ is the death benefit at time s per 1- unit face amount,[3] $f(s)$ is the probability density function of the future lifetime random variable S at age x: ${}_s p_x \mu_{x+s}$, in which μ_{x+s} is the force of mortality at age of x+s and ${}_s p_x$ denotes the probability of a person at age x who will survive s years.

Note that $b(s)$=1 for traditional whole life insurance, which means that the death benefit is fixed at the face amount of the policy. In such a case, equation (1) implies that the net single premium is a function of the force of interest δ with respect to $f(s)$. Should mortality rise unexpectedly, the collected premium has insufficient time to accumulate to the death benefit; on the other hand, the policyholders as a whole pay

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[2]The Laplace transform $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-\delta t} f(t)dt$ is an integral transform widely used in mathematics with many applications in physics and engineering. It turns convolution into multiplication, the latter being easier to solve because of its algebraic form.

[3]

too much if mortality improves more than expected. The product makes the insurer and insured subject to the mortality risk. Can we mitigate such risk through product design?

One design is to make the death benefit an increasing function of death time so that the accumulated value of the premium can match the death benefit no matter how mortality varies. The Laplace transform format sheds light on a possible solution: changing the parameter of $\phi(.)$ as

$$\phi(\delta-\gamma)=\int_0^\infty e^{-\delta s}e^{\gamma s}\,{}_s p_x\,\mu_{x+s}\,ds=\mathcal{L}\left\{e^{\gamma s}f(s)\right\}. \tag{2}$$

The product implied by equation (2) is an increasing whole life insurance policy. Its death benefit increases continuously at the annual rate of γ. The parameter γ controls how much to pay while δ would reflect the time value of payment. Appropriate choices on (γ, δ), such as $\gamma\approx\delta$, can make the present value of the death benefit insensitive to the timing of death that in turn is affected by mortality. Such design can thus mitigate the mortality risk.

2.2 Formal Development

We elaborate the above idea by examining the expected reserve of the appropriately calibrated whole life insurance product to see whether it can be immunized from the mortality risk by itself. When a new policy of the calibrated product is sold now (i.e., at time 0) to a customer at age *x* with face amount 1 (without loss of generality), the expected reserve at time *t* of the policy is as in equation (3):

$${}_tV_x=\int_0^\infty F_0e^{\gamma t}e^{\gamma s}e^{-\delta s}\,{}_s p_{x+t}\,\mu_{x+t+s}\,ds\ . \tag{3}$$

Referring to the derivation in Bowers et al (1997, chapter 4), we may decompose ${}_tV_x$ into two parts:

$$\begin{aligned}{}_tV_x&=\mathrm{E}\left[F_0e^{\gamma(t+S)}e^{-\delta S}\right]\\&=\mathrm{E}\left[F_0e^{\gamma(t+S)}e^{-\delta S}\,|\,S\le h\right]\Pr\left(S\le h\right)+\mathrm{E}\left[F_0e^{\gamma(t+S)}e^{-\delta S}\,|\,S>h\right]\Pr\left(S>h\right),\end{aligned} \tag{4}$$

where $\Pr\left(S\le h\right)={}_hq_{x+t}$ and $\Pr\left(S>h\right)={}_hp_{x+t}$.

Since the conditional p.d.f. of *S* given *S*≤*h* is

$$f(s \mid S \le h) = \begin{cases} \dfrac{f(s)}{F(h)} = \dfrac{{}_s p_{x+t}\ \mu_{x+t+s}}{{}_h q_{x+t}} & 0 \le s \le h \\ 0 & elsewhere \end{cases},$$

$$\mathrm{E}\left[F_0 e^{\gamma(t+S)} e^{-\delta S} \mid S \le h\right] = \int_0^h F_0 e^{\gamma(t+s)} e^{-\delta s} \frac{{}_s p_{x+t}\ \mu_{x+t+s}}{{}_h q_{x+t}}\, ds \tag{5}$$

and

$$\begin{aligned} \mathrm{E}\left[F_0 e^{\gamma(t+S)} e^{-\delta S} \mid S > h\right] &= e^{\gamma h}\,\mathrm{E}\left[F_0 e^{\gamma(t+S-h)} e^{-\delta(S-h)} \mid (S-h) > 0\right] \\ &= e^{\gamma h} e^{-\delta h}\,\mathrm{E}\left[F_0 e^{\gamma(t+S)} e^{-\delta S} \mid (S-h) > 0\right] = e^{\gamma h} e^{-\delta h}\,{}_t V_{x+h}. \end{aligned} \tag{6}$$

Substituting equations (5) and (6) into (4) yields

$${}_t V_x = \int_0^h e^{\gamma(t+s)} e^{-\delta s} \frac{{}_s p_{x+t}\ \mu_{x+t+s}}{{}_h q_{x+t}}\, ds\ {}_h q_{x+t} + e^{\gamma h} e^{-\delta h}\,{}_t V_{x+h}\ {}_h p_{x+t}. \tag{7}$$

Multiplying both sides of equation (7) by -1, adding ${}_t V_{x+h}$, and then dividing by *h*, we obtain

$$\frac{{}_t V_{x+h} - {}_t V_x}{h} = \frac{-1}{h}\int_0^h e^{\gamma(t+s)} e^{-\delta s}\,{}_s p_{x+t}\ \mu_{x+t+s}\, ds + {}_t V_{x+h} \frac{\left(1 - e^{\gamma h} e^{-\delta h}\,{}_h p_{x+t}\right)}{h}. \tag{8}$$

The limits of the two items on the right-hand side of equation (8) are as follows:

$$\begin{aligned} \lim_{h\to 0} \frac{-1}{h}\int_0^h e^{\gamma(t+s)} e^{-\delta s}\,{}_s p_{x+t}\ \mu_{x+t+s}\, ds &= -\frac{d}{ds}\int_0^s e^{\gamma(t+s)} e^{-\delta s}\,{}_s p_{x+t}\ \mu_{x+t+s}\, ds \,\Big|_{s=0} \\ &= -e^{\gamma(t+s)} e^{-\delta s}\,{}_s p_{x+t}\ \mu_{x+t+s}\Big|_{s=0} = -e^{\gamma t}\mu_{x+t}, \end{aligned} \tag{9}$$

and

$$\begin{aligned} \lim_{h\to 0} {}_t V_{x+h} \frac{\left(1 - e^{\gamma s} e^{-\delta s}\,{}_s p_{x+t}\right)}{h} &= {}_t V_x \frac{d}{ds}\left(1 - e^{\gamma s} e^{-\delta s}\,{}_s p_{x+t}\right)\Big|_{s=0} \\ &= {}_t V_x \left(e^{\gamma s} e^{-\delta s}\,{}_s p_{x+t}\mu_{x+t+s}\Big|_{s=0} + (\delta - \gamma) e^{\gamma s} e^{-\delta s}\,{}_s p_{x+t}\Big|_{s=0}\right) \\ &= {}_t V_x \left(\mu_{x+t} + (\delta - \gamma)\right) \end{aligned}. \tag{10}$$

The limit of equation (8) is thus

$$-e^{\gamma t}\mu_{x+t} + {}_tV_x\left(\mu_{x+t}+\delta-\gamma\right). \tag{11}$$

The above derivation means that

$$\frac{d}{dx}{}_tV_x = -e^{\gamma t}\mu_{x+t} + {}_tV_x\left(\mu_{x+t}+\delta-\gamma\right). \tag{12}$$

Rearranging equation (12), we get

$${}_tV_x = \frac{\frac{d}{dx}{}_tV_x + e^{\gamma t}\mu_{x+t}}{\mu_{x+t}+\delta-\gamma}. \tag{13}$$

Note that $e^{\gamma t}\mu_{x+t}$ is always positive since $\mu_{x+t} > 0$. Then we have the following boundary conditions of the expected reserves:

1. when $\mu_{x+t}+\delta-\gamma > 0$ and $\frac{d}{dx}{}_tV_x \geq 0$, ${}_tV_x \geq \frac{e^{\gamma t}\mu_{x+t}}{\mu_{x+t}+\delta-\gamma}$,

2. when $\mu_{x+t}+\delta-\gamma < 0$ and $\frac{d}{dx}{}_tV_x \geq 0$, ${}_tV_x \leq \frac{e^{\gamma t}\mu_{x+t}}{\mu_{x+t}+\delta-\gamma}$,

3. when $\mu_{x+t}+\delta-\gamma > 0$and $\frac{d}{dx}{}_tV_x \leq 0$, ${}_tV_x \leq \frac{e^{\gamma t}\mu_{x+t}}{\mu_{x+t}+\delta-\gamma}$, and

4. when $\mu_{x+t}+\delta-\gamma < 0$and $\frac{d}{dx}{}_tV_x \leq 0$, ${}_tV_x \geq \frac{e^{\gamma t}\mu_{x+t}}{\mu_{x+t}+\delta-\gamma}$.

The equation of the boundary can be written as:

$$\left(\mu_{x+t}+\delta-\gamma\right)\left({}_tV_x - e^{\gamma t}\right) = -e^{\gamma t}\left(\delta-\gamma\right). \tag{14}$$

As Figures 1 and 2 display, Equation (14) represent hyperbolas centered at $(-\delta+\gamma, e^{\gamma t})$ on the $(\mu_{x+t}, {}_tV_x)$ plane. The two asymptotes of the hyperbola are given by $\mu_{x+t} = -\delta+\gamma$ and ${}_tV_x = e^{\gamma t}$. The hyperbola with positive $(\delta - \gamma)$ lays in the 2nd and 4th quadrants coordinated with respect to the center while the hyperbola with negative $(\delta - \gamma)$ lays in the 1st and 3rd quadrants as shown by Figure 1 and Figure 2 respectively. The hyperbola in Figure 1 exhibit positive slopes while that in Figure 2 have negative slopes.

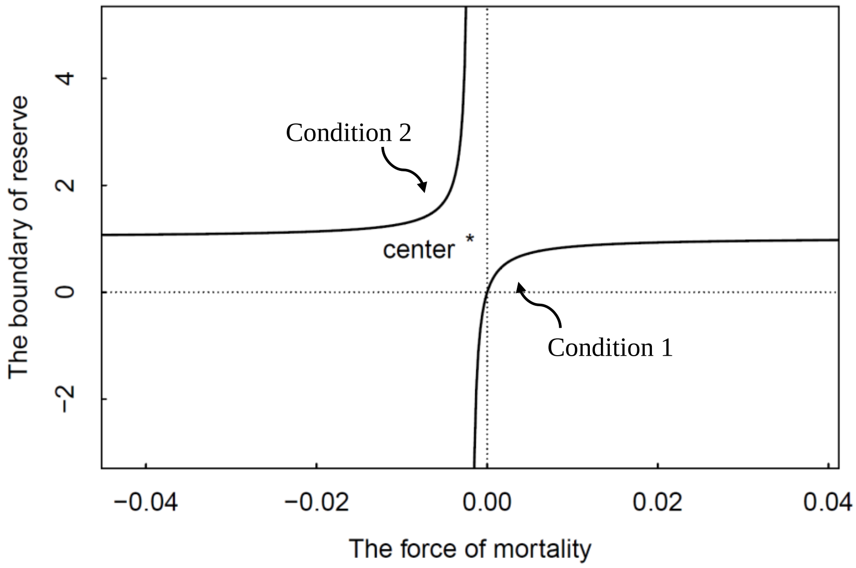

Figure 1 The boundary of the expected reserve with positive ($\delta - \gamma$) on the (μ_{x+t}, ${}_tV_x$) plane

Note: We are aware that negative $\mu_{\text{x+t}}$ is not reasonable but retain them to show a complete hyperbola.

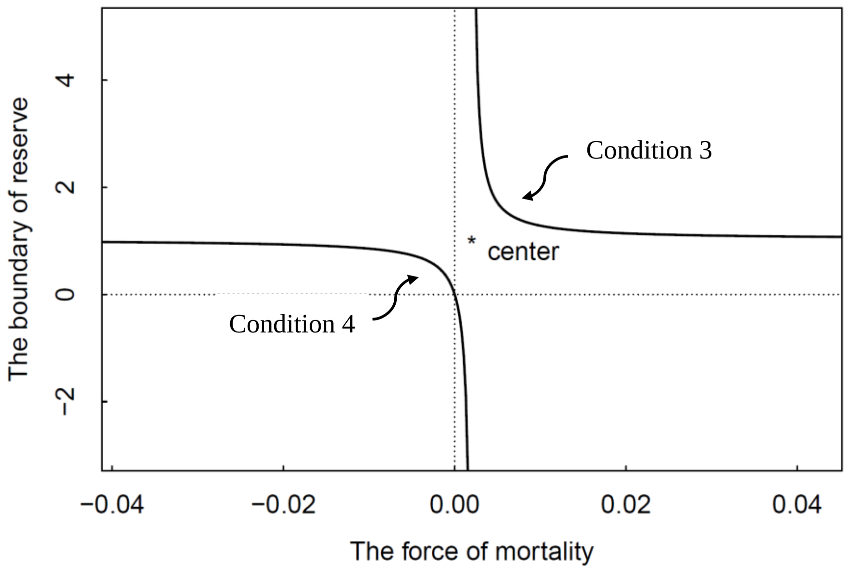

Figure 2 The boundary reserve with negative ($\delta - \gamma$) on (μ_{x+t}, ${}_tV_x$) plane

In Figure 1, the curve on the down-right of the center matches the criteria of the condition 1. Assuming that μ_{x+t} is a increasing function[4] of age x, with criterion $\frac{d}{dx}{}_tV_x \geq 0$ of condition 1, we may deduce that the ${}_tV_x$ is increasing along with μ_{x+t}, and with criterion $\mu_{x+t}+\delta-\gamma > 0$, the lower bound of expected reserve in condition 1 should be on the down-right in respect to the center of hyperbola. The up-left curve in Figure 1 matches the criteria of the condition 2. With criteria $\mu_{x+t}+\delta-\gamma < 0$ and $\frac{d}{dx}{}_tV_x \geq 0$, note that the condition 2 shown on the down-left part of the hyperbola does not exist in reality since the force of mortality μ_{x+t} should not be negative.

Following the same deduction, we identify the hyperbola with negative $(\delta - \gamma)$ in Figure 2, of which the up-right curve represents the upper bound of the expected reserve in the condition 3 and the down-left curve represents the lower bound of the expected reserve in the condition 4. The feature of boundary reserve in condition 3 and 4 is decreasing along with μ_{x+t}. The expected reserve may be more/less than or equal to the lower/upper bound of expected reserve in practice.

When narrowing down the possibility of $(\delta - \gamma)$ in the four conditions we find out the boundary reserve is equal to the expected reserve if $(\delta - \gamma) = 0$. When $(\delta - \gamma)$ is equal to 0, the lower/upper bound of the expected reserve in Figure 1 is folded together with the upper/lower bound of that in Figure 2. The feature of the special case with $(\delta - \gamma) = 0$ turns into a horizontal line as shown in Figure 3 which is exactly one of the asymptotes of the hyperbola ${}_tV_x = e^{\gamma t}$.

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[4] $\frac{d}{dx}{}_tV_x = \frac{d\,{}_tV_x}{d\mu_{x+t}} \cdot \frac{d\mu_{x+t}}{dx}$

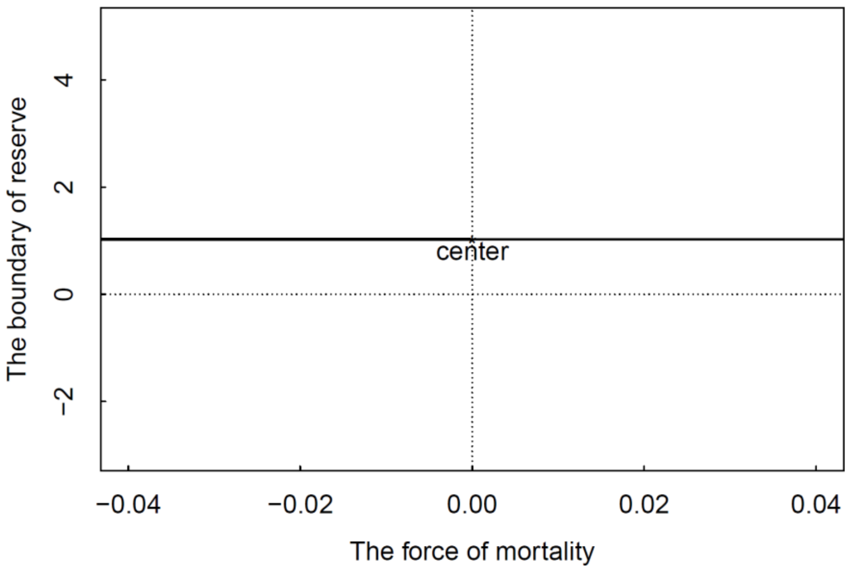

Figure 3 The boundary reserve with $(\delta - \gamma) = 0$ on the $(\mu_{x+t}, {}_tV_x)$ plane

Further Analyses on the $(x+t, {}_tV_x)$ plane

Mortality rate risk exists because we evaluate risk according to ages. If insurance companies price life insurance products according to real aging situation, the companies may not have mortality rate risk. In fact, neither can we observe aging of individuals directly, nor can we price the mortality risk by each individual's aging situation. What we can observe in nature is individual age and mortality rate with respect to age statistically. Then, the force of mortality is quantified to express aging with respect to age in methodology. Since we price insurance product based on ages, not aging, mortality rate risk emerges accordingly. Thus, acquiring the information on $(x+t,\ {}_tV_x)$ plane is important in determining the mortality rate risk of the specified product along with the changes of mortality rate.

Next, we intend to demonstrate the relationship of expected reserve and age by the feature of the boundary reserve ${}_tV_x = \dfrac{e^{\gamma t}\mu_{x+t}}{\mu_{x+t} + \delta - \gamma}$ coordinated on the $(x+t\ , {}_tV_x)$ plane.

With the information on the $(\mu_{x+t}, {}_tV_x)$ plane in last section, we can transform the feature of the boundary reserve ${}_tV_x = \dfrac{e^{\gamma t}\mu_{x+t}}{\mu_{x+t}+\delta-\gamma}$ with respect to the force of mortality into the feature of that with respect to age by assuming that the force of mortality μ_{x+t} is a monotonic increasing function of age $x+t$. We then can illustrate features of the boundary reserve according to age $x+t$ on the $(x+t\ ,\ {}_tV_x)$ plane and explore the relationship of expected reserve and age.

In case of the positive $(\delta-\gamma)$, we transform the feature of the boundary reserve on the $(\mu_{x+t},\ {}_tV_x)$ plane in Figure 1 into the feature on the $(x+t\ ,\ {}_tV_x)$ plane as shown in Figure 4. Also, in case of the negative $(\delta-\gamma)$, we transform the feature of the boundary reserve in Figure 2 into the feature on the $(x+t\ ,\ {}_tV_x)$ plane as shown in Figure 5. The negative values of μ_{x+t} on the $(\mu_{x+t},\ {}_tV_x)$ plane cannot exhibit on the $(x+t\ ,\ {}_tV_x)$ plane for each age is of positive force of mortality.

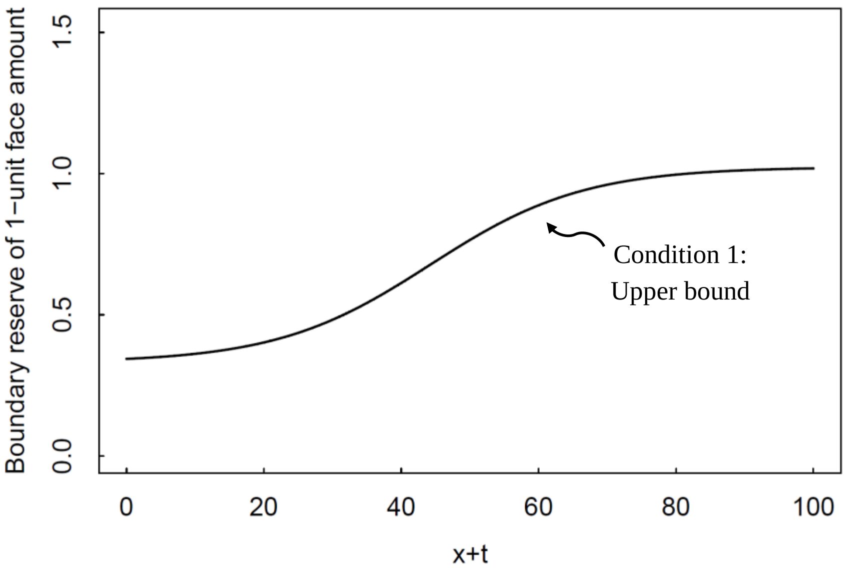

Figure 4 The boundary reserve with positive $(\delta-\gamma)$ on $(x+t,\ {}_tV_x)$ plane

Note that the illustration is using the Makeham model $\mu_{x+t} = 9.566\times10^{-4} + 5.162\times10^{-5}\times1.09369^{x+t}$, which is cited from Melnikov and Romaniuk (2006) and the original data is based on the mortality rates from 1959 to 1999 in American (Pollard, 1973).

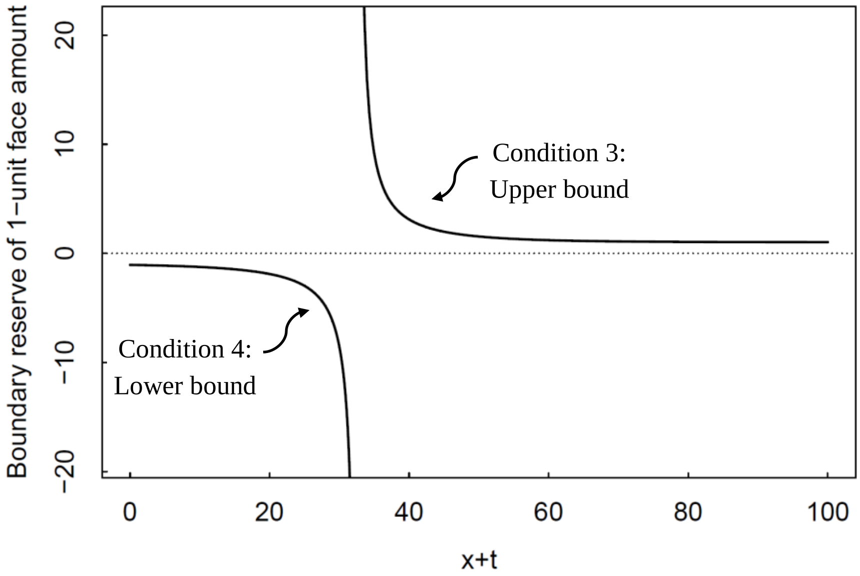

Figure 5 The boundary reserve with negative ($\delta - \gamma$) on (x+t, ${}_tV_x$) plane

Note that the illustration is using the Makeham model $\mu_{x+t} = 9.566 \times 10^{-4} + 5.162 \times 10^{-5} \times 1.09369^{x+t}$, which is cited from Melnikov and Romaniuk (2006) and the original data is based on the mortality rates from 1959 to 1999 in American (Pollard, 1973).

We demonstrate the four conditions of equation (13) in corresponding features on the (x+t ,${}_tV_x$) plane. The feature of the boundary reserve in Figure 4 shows the ages with positive value of μ_{x+t} which is the case in condition 1. The case in condition 2 does not show out on the (x+t ,${}_tV_x$) plane for the value of μ_{x+t} is negative. In Figure 5, the feature of boundary reserve on the up-right part is the case in condition 3 that follows the criterion of $\mu_{x+t}+\delta-\gamma>0$; while the other on the down-left part is the case in condition 4 that follows the criterion of $\mu_{x+t}+\delta-\gamma<0$ on the (x+t , ${}_tV_x$) plane.

The feature of the boundary reserve in Figure 4 appears faster increasing than that in 1st quadrant of Figure 1 along with the horizontal axis. As the horizontal axis of μ_{x+t} is changed into that of the age x+t, the horizontal axis is re-scaled evenly at the measurement of age on the (x+t , ${}_tV_x$) plane. The slope $\frac{d}{dx}{}_tV_x$ of elderly ages in Figure 4 is more than that $\frac{d\,{}_tV_x}{d\,\mu_{x+t}}$ in figure 1 since $\frac{d}{dx}{}_tV_x = \frac{d\,{}_tV_x}{d\,\mu_{x+t}}\frac{d\,\mu_{x+t}}{dx}$ and $\frac{d\,\mu_{x+t}}{dx}$ of the elderly age is much higher than that of the young ages.

In Figure 4, the boundary reserve of specified product with positive ($\delta - \gamma$), appears that the risk rooting in “when to pay” is more than that rooting in “how much to pay”. As for our death benefit product, the risk rooting in “when to pay” represents mortality risk because when the realized mortality rate rises up, it means that more than expected people of cohorts are paid earlier than expected timing. The insurance companies get loss by the time value of extra death benefit claim due to increase of mortality rate. The risk rooting in “how much to pay” in our design is that we set the increment of death benefit each time that is only provided to the survivors at the time. The risk rooting in “how much to pay” is a form of longevity risk. The mortality risk exposure is more than longevity exposure in Figure 4. In general, most life insurance products are grouped in this type of feature in Figure 4 as the mortality risk is more than the longevity risk.

The feature in Figure 5 is seemingly but not exact hyperbola on the $(x+t\ ,\ {}_tV_x)$ plane. The age around 33 years old is a critical age that the boundary reserve is unbounded. The boundary reserve in the case of negative $(\delta - \gamma)$ is decreasing along with age more than 33 years old. When $(\delta - \gamma)$ is negative, we may say that the risk rooting in “how much to pay” in our design is more than that rooting in “when to pay”. That is to say the longevity risk of the case is more than mortality risk following the similar elaboration in the case of positive $(\delta - \gamma)$ above. This type of insurance product is rare. But it ever shortly appeared in Taiwan insurance market. The companies designed an increasing whole life insurance with $\gamma > \delta$. For instance, the death benefit is based on face amount compounded by 4% annually and annual interest rate is 2.5%.

As for the special case on the $(\mu_{x+t},\ {}_tV_x)$ plane in Figure 3, we transform the horizontal axis to age $x+t$ and display the case of $(\delta - \gamma)=0$ on $(x+t,\ {}_tV_x)$ plane as shown in Figure 6. The upper bound is overlapped with the lower bound of the expected reserve. The feature of the boundary reserve is a horizontal line that is the same as in Figure 3. The difference of features between Figure 3 and Figure 6 is the measurement of horizontal axis that cannot alter the shape of horizontal line. The function of the special horizontal line ${}_tV_x = e^{\gamma t}$ on (x+t, ${}_tV_x$) plane is the same as that on the (μ_{x+t}, ${}_tV_x$) plane since the case is irrelevant to the variable of horizontal axis. In this special case, the feature is not just the boundary reverse but also the expected reserve. We then come out an only solution that fulfills the criteria of the four conditions of equation (13) represented the relationship between expected reserve and age.

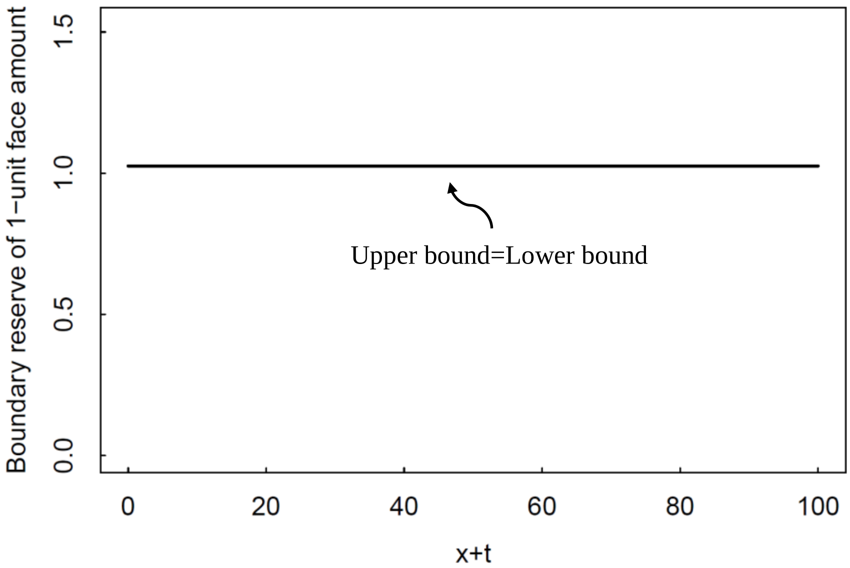

Figure 6 The boundary reserve with $\delta - \gamma = 0$ on (x+t , ${}_tV_x$) plane

Note that the illustration is using the Makeham model $\mu_{x+t} = 9.566\times10^{-4} + 5.162\times10^{-5}\times1.09369^{x+t}$, which is cited from Melnikov and Romaniuk (2006) and the original data is based on the mortality rates from 1959 to 1999 in American (Pollard, 1973).

The expected reserve is a horizontal line on (x+t, ${}_tV_x$) plane reveals that the expected reserve is irrelevant to age. We may make use of this property on risk mitigation within product. When we design such a specified product with $(\delta - \gamma)$=0, the risk rooting in “when to pay” is exactly equal to the risk rooting in “how much to pay”. The mortality risk is mitigated totally by the longevity risk for each age in the case. Furthermore, the expected reserve is level to all ages and to the force of mortality of each age. The expected reserve of the policy will not be affected by the changes of mortality rate and will be equal to the realized reserve. As the realized reserve remains the same as the expected reserve with the changes of mortality rate, the risk is immunized within the policy.

3. Numerical illustrations

The basic assumptions are set up in Table 1. We assume that the face amount is US$100,000 for the specified whole life insurance and the premium is paid by single premium. Assume that the company only sells the specified whole life insurance product with given value of δ. The natural hedging strategy depends on the policy design of the force of amount γ. γ can be any given value in a reasonable risk mitigation strategy.

Table 1

Basic assumption for the new form of the whole life insurance product

Age of insured	25, 45
Gender	Male
Face amount	100,000
The initial value of force of interest rate (δ)	4%
Death benefit	100,000 compounded by $\gamma(t)$
Benefit period	Whole life
Method of paying premium	Single premium

3.1 The optimal strategy with $\gamma = \delta$

We first investigate the case with the value of γ equal to the value of δ. Our pricing mortality rate is based on the Makeham model (Melnikov and Romaniuk, 2006). We take the 5^{th} policy year as an example to examine the mortality rate risk which is the difference of the realized reserve and the expected reserve. The realized reserve is evaluated by 20% up shock or 20% down shock of the mortality rate. Keeping the setting of δ and γ to satisfy the equation $(\delta - \gamma) = 0$ in all time, we can offer three different types of the designed product as the following examples.

Product Design 1: The product is a type of traditional increasing whole life insurance.

(a). With δ = constant number

Let $\gamma = \delta$ = constant = 4% through the whole policy year. The death benefit is compound at γ continuously and it is indicated as the following equation

$F_t = F_0 \exp(\gamma t)$,

where F_0 is the face amount, t is the policy year. The outcome is shown in the Table 2.

Table 2 The Liability at the End of the 5th Policy Year of Illustrated Insurance Product for Different Mortality Bases

			$\gamma(=4\%) = \delta(=4\%)$		
	(1)	(2)	(3)	(4)=[(2)-(1)]/(1)	(5)=[(3)-(1)]/(1)
age	Basis	20% Up Shock	20% Down Shock	Reserve Changed	Reserve Changed
25	122,140	122,140	122,140	0%	0%
45	122,140	122,140	122,140	0%	0%

.

As we can see in Table 2, the reserve remains unchanged after the shock of mortality rate. The result is consistent with the case of $(\delta - \gamma) = 0$ in our theoretical analysis. Thus, the mortality rate risk of the specified product is none since the risk exposure does not increase or decrease caused by the 20% shock at the 5th policy year. The specified whole life insurance product appears no mortality risk or longevity risk in all ages.

The product design 1 is a traditional increasing whole life insurance. If the interest rate keeps the same as the force of amount γ, the product fulfills the criteria $\delta = \gamma$ and there is no mortality rate risk at all. While in reality, the interest rate is not fixed along with the whole policy years, the product may be exposed to risks.

(b). δ obtained by CIR model with a constant long term mean

To capture the risks in reality at the beginning of the policy, we establish an interest rate model of one factor CIR[5] model and mortality rate model of LC[6] (Lee & Carter, 1992) model to capture the dynamic of interest rate and mortality in the whole policy years. The γ remains constant in the product design. All numerical results are obtained with 10,000 simulations. The product design 1 of dynamic interest rate (δ) with a constant long term mean that is equal to γ. We display the results of γ = 6.5% and the other comparison panel of γ = 0% as shown in the Table 3.

Table 3 The Liability at the End of the 5th Policy Year of Illustrated Insurance Product with CIR Model and LC Model for Different Mortality Bases

Panel A: γ (=6.5%) = Long Term Mean of Interest Rate Model					
	(1)	(2)	(3)	(4)=[(2)-(1)]/(1)	(5)=[(3)-(1)]/(1)
age	Basis	Mortality Rate 20% Up Shock	Mortality Rate 20% Down Shock	Reserve Changed	Reserve Changed
25	153,363	152,143	153,547	-0.80%	0.12%
45	146,222	145,267	146,424	-0.65%	0.14%
Panel B: γ (= 0%) ≠ Long Term Mean of Interest Rate Model					
	(1)	(2)	(3)	(4)=[(2)-(1)]/(1)	(5)=[(3)-(1)]/(1)
age	Basis	Mortality Rate 20% Up Shock	Mortality Rate 20% Down Shock	Reserve Changed	Reserve Changed
25	6,378	7,133	5,536	11.84%	-13.20%
45	17,509	19,195	15,570	9.63%	-11.07%

The results in Table 3 that shows mitigation of risk in Panel A is better than those in Panel B. Even though that the product design 1 in Panel A is not exactly keeping $\gamma(t) = \delta(t)$ point to point, it still better off the results with $\gamma(t)=0$.

Product Design 2: The type of product is an interest rate variable whole life insurance.

In order to keep the criteria of $\gamma(t) = \delta(t)$ during the same time period, t =1,

[5] $dr(t) = \kappa(\theta - r(t))dt + \sigma\sqrt{r(t)}dW(t)$, with κ = 0.25, θ = 0.065, σ = 0.07 (Liu (2013), Chan et al. (1992))

[6] $\log m(x, t) = a_x + b_x k_t + \varepsilon_{xt}$, The index k_t is modeled by a random walk with drift term: $k_t = k_{t-1} + c + e_t$. The data is US mortality rates from 1961 to 2010 obtained on www.mortality.org. The parameter estimates for LC model are: c = -1.066826, σ_e = 1.534088

2…etc. We decide the $\gamma(t)$ immediately after we obtain a new $\delta(t)$ in the market each time. We let $\gamma(t)$ as close to $\delta(t)$ as possible. The $\delta(t)$ is piecewise continuously along with time t and so is the $\gamma(t)$.We assume $\delta(1) = \delta$, $\gamma(1) = \delta(1)$ for the first policy year. From the second policy year on, we declare a new interest rate at the beginning of each policy year that generates a new force of interest rate δ(t). Let $\gamma(\text{t}) = \gamma_t = \delta(\text{t})$. We obtain the indication of the death benefit this kind of policy is

$$F_t = F_0 \exp(\Sigma\gamma_t).$$

As the interest rate is variable at each policy year, we assume the interest rate scenarios for the 2^{nd} to 5^{th} policy year. The values of the force of interest rate for the first five policy years are as shown in Table 4.

Table 4 The Scenarios of the Values of the Force of Interest Rate for the First Five Policy Years.

Policy Year	1	2	3	4	5
Scenario A	δ=4%	δ(2)=4.25%	δ(3)=4.50%	δ(4)=4.50%	δ(5)=4.75%
Scenario B	δ=4%	δ(2)=3.75%	δ(3)=3.75%	δ(4)=3.50%	δ(5)=3%

The outcome of product design 2 is shown in Table 5. We can see that the values of reserve are not changed by the shock of mortality rate in Table 5. The mortality risk and longevity risk indicated in column (4) and column (5), respectively, are shown no risk by the changes of mortality rate because the total mortality rate risk is immunized within the policy.

Table 5 The Liability at the End of the 5^{th} Policy Year of Product Design 2 for Different Mortality Bases

		$\gamma(\text{t}) = \delta(\text{t})$				
		(1)	(2)	(3)	(4)=[(2)-(1)]/(1)	(5)=[(3)-(1)]/(1)
Scenario	age	Basis	20% Up Shock	20% Down Shock	Reserve Changed	Reserve Changed
A	25	124,608	124,608	124,608	0%	0%
	45	124,608	124,608	124,608	0%	0%
B	25	119,722	119,722	119,722	0%	0%
	45	119,722	119,722	119,722	0%	0%

The product includes a level benefit whole insurance and an extra insurance benefit (i.e. dividend or increment of death benefit) that depends on the declaration of interest rate each policy year. The interest rate variable life insurance in the United States is one of the kinds in this product group. Our product design is based on the interest rate variable life insurance and keeping the criterion $\gamma(t)=\delta(t)$. The difference from that the interest rate variable insurance product provides dividend to policyholders, the product 2 provides the increment of death benefit. Each increment of death benefit in our product should be the same as the dividend of interest rate variable insurance product generated by the declared interest rate. As long as the product meets the criterion $\gamma(t)=\delta(t)$, whether the extra benefit is called either the increment of death benefit in our design or the dividend in the content of interest rate variable life insurance, it does not matter the achievement of risk mitigation. In our product design here, we take the extra death benefit increased by each declared interest rate and provide no dividend. As to fulfill the criteria, the design of interest rate variable life insurance should take death benefit and dividends into account on assessment of risk. Since the interest rate is declared in related to market interest rate, the type of product declines the threat of interest rate risk comparing to the product with fixed interest rate.

Product Design 3: The interest rate variable increasing whole life insurance.

The idea of this product is similar to Product design 2 that the decision of $\gamma(t)$ is soon after the newest $\delta(t)$ we can get in the market. Let $\gamma(t) = \gamma_0 + \Delta\gamma_t = \delta(t)$, during the same time period, t =1, 2…etc. Set the initial value of the force of amount $\gamma_0 = 2\% < \delta = 4\%$ at time 0. Let $\delta(1) = \delta$, $\gamma(1) = \gamma_0 + \Delta\gamma_1 = \delta(1)$ in the first policy year. From the 2nd policy year on, declare a new interest rate in each following policy year that generate a new force of interest rate $\delta(t)$. As $\gamma(t) = \gamma_0 + \Delta\gamma_t = \delta(t)$, the death benefit is

$$F_t = F_0 \exp(\gamma_0 t + \Sigma\Delta\gamma_t) = F_0 \exp(\gamma_0 t) + F_0 \exp(\Sigma\Delta\gamma_t).$$

The product is a combination of product 1 and product 2. The product contains a fixed increment death benefit that makes this part of product looks like a traditional increasing whole life and a variant increment death benefit that makes this part of product seems like an interest rate variable life insurance. The variant increment death benefit is like product design 2 determined by the declaration of interest rate. We assume the interest rate scenarios for the 2nd to 5th policy year. The values of the force of interest rate for the first five policy years are as shown in Table 6.

Table 6 The Scenarios of the Values of the Force of Interest Rate for the First Five Policy Years

Policy Year	1	2	3	4	5
Scenario C	δ=4%	$\delta(2)$=4%	$\delta(3)$=4.50%	$\delta(4)$=4.50%	$\delta(5)$=4.50%
	$\Delta\gamma_1$=2%	$\Delta\gamma_2$=2.25%	$\Delta\gamma_3$=2.50%	$\Delta\gamma_4$=2.50%	$\Delta\gamma_5$=2.75%
Scenario D	δ=4%	$\delta(2)$=3.50%	$\delta(3)$=3.50%	$\delta(4)$=3.50%	$\delta(5)$=3.25%
	$\Delta\gamma_1$=2%	$\Delta\gamma_2$=1.75%	$\Delta\gamma_3$=1.75%	$\Delta\gamma_4$=1.50%	$\Delta\gamma_5$=1%

The outcome of product 3 is shown in Table 7. We can also see that the values of reserve are not changed by the shock of mortality rate in Table 7. The mortality risk and longevity risk indicated in column (4) and column (5), respectively, are shown no risk by the changes of mortality rate because the total mortality rate risk is immunized within the policy.

Table 7 The Liability at the End of the 5th Policy Year of Product 3 for Different Mortality Bases

			$\gamma(t) = \gamma_0 + \Delta\gamma_t = \delta(t)$			
		(1)	(2)	(3)	(4)=[(2)-(1)]/(1)	(5)=[(3)-(1)]/(1)
Scenario	age	Basis	20% Up Shock	20% Down Shock	Reserve Changed	Reserve Changed
C	25	123,986	123,986	123,986	0%	0%
	45	123,986	123,986	123,986	0%	0%
D	25	119,423	119,423	119,423	0%	0%
	45	119,423	119,423	119,423	0%	0%

3.2. The secondary strategy with $0<\gamma<\delta$[7]

When the strategy is with $\gamma<\delta$, our objective is to diminish the mortality rate risk, not to immunize the risk. As we elaborate in this article, the risk rooting in “when to pay” of a death benefit life insurance is mortality risk along with the changes of mortality rate, and the risk rooting in “how much to pay” of a death benefit life insurance is longevity risk. We take a death benefit whole life insurance as an example. The outcome of the changes of the mortality rate with 20% shock is in Table 8.

Table 8 The Liability at the End of the 5th Policy Year of Illustrated Insurance Product for Different Mortality Bases

Panel A: γ (=2%) <δ (=4%)					
	(1)	(2)	(3)	(4)=[(2)-(1)]/(1)	(5)=[(3)-(1)]/(1)
age	Basis	20% Up Shock	20%Down Shock	Reserve Changed	Reserve Changed
25	45,368	47,287	43,104	4.230%	-4.990%
45	63,758	66,020	61,018	3.548%	-4.297%
Panel B: γ (=0%) <δ (=4%)					
25	18,612	20,153	16,869	8.280%	-9.350%
45	35,216	37,608	32,420	6.792%	-7.940%

The product in Panel A with δ, 4% and γ, 2% is compare to that in Panel B with δ, 4% and γ, 0%. The product in Panel A is exposed to both the mortality risk and the longevity risk and the longevity risk is less than the mortality risk. The product in Panel B is exposed to mortality risk only. When the mortality rate is changed by 20% shock, the expected reserve of both products is changed. With increase of reserve as indicated in column 4 with respect to 20% up shock of mortality rate, product in Panel A is exposed to risk of 3%~4% more than that in basis mortality rate. And the product in Panel B is exposed to that of 6~8% more than that in basis mortality rate by the changes of mortality rate. The secondary strategy is to diminish the risk within the policy by creating longevity risk in a life insurance product to lower the exposure of mortality risk and the strategy helps us to design such products upon the demand of risk exposure.

[7] With product design of $\gamma>\delta$, the life insurance product becomes exposed to the longevity risk only. That is not normal in a pure death benefit life insurance product. In the normal situation, insurance companies may not design such products to confuse themselves in identification of the longevity risk resulting from a pure death benefit life insurance.

4. Annuity

We may apply the same technique with the force of amount γ in section 2 for a whole life annuity product of 1 unit face amount per annum payable continuously while (x) survives. Since the annuity product is paying benefit as long as the insured survives, it is exposed to longevity risk of "when to stop paying." The term is the same as "when to pay" of life insurance product is the time value determined by δ. We are adding the force of amount γ in product structure to create a mortality risk of "how much to pay" to mitigate the longevity risk.

4.1 Theoretical development on annuity product

Thus, we set the function of survivor benefit for the annuity product, $b''(s)= e^{-\gamma s}$, per 1- unit face amount at time s, where $\gamma >0$. With S representing the future lifetime of (x), the present value of the annuity payments made up until death is

$$Y = \int_0^S e^{-\gamma u} e^{-\delta u}\, du = \frac{1-e^{-(\delta+\gamma)S}}{\delta+\gamma}. \qquad (14)$$

The net single premium of the specified annuity product is denoted as

$$\begin{aligned}\int_0^\infty Y\ {}_s p_x\ \mu_{x+s}\, ds &= \int_0^\infty \frac{1-e^{-(\delta+\gamma)s}}{\delta+\gamma}\ {}_s p_x\ \mu_{x+s}\, ds \\ &= \frac{1}{\delta+\gamma} - \frac{1}{\delta+\gamma}\int_0^\infty e^{-\gamma s} e^{-\delta s}\ {}_s p_x\ \mu_{x+s}\, ds\end{aligned}. \qquad (15)$$

Integration by parts, the equation (15) can also be expressed as

$$\begin{aligned}\int_0^\infty Y\ {}_s p_x\ \mu_{x+s}\, ds &= \frac{1}{\delta+\gamma} - \frac{1}{\delta+\gamma}\left(e^{-rs} e^{-\delta s}\big|_{s=0} + \int_0^\infty -(\delta+\gamma) e^{-rs} e^{-\delta s}\ {}_s P_x\, ds \right) \\ &= \int_0^\infty e^{-rs} e^{-\delta s}\ {}_s P_x\, ds\end{aligned}. \qquad (16)$$

When the annuity product is sold to a customer at age x with 1-unit face amount, t policy year passing by, the expected reserve at the end of t policy year for this annuity policy can be expressed as equation (17) or equation (18)

$$ {}_tV_x^a = \frac{e^{-rt}}{\delta+\gamma} - \frac{1}{\delta+\gamma}\int_0^\infty e^{-rt}e^{-rs}e^{-\delta s}{}_sP_{x+t}\,\mu_{x+t+s}\,ds, \tag{17} $$

or

$$ {}_tV_x^a = \int_0^\infty e^{-rt}e^{-rs}e^{-\delta s}{}_sP_{x+t}\,ds. \tag{18} $$

Having equation (17) differentiated by *x* and applying equation (10), we obtain

$$ \begin{aligned} \frac{d}{dx}{}_tV_x^a &= \frac{d}{dx}\left(\frac{e^{-rt}}{\delta+\gamma} - \frac{1}{\delta+\gamma}\int_0^\infty e^{-rt}e^{-rs}e^{-\delta s}{}_sP_{x+t}\,\mu_{x+t+s}\,ds\right) \\ &= -\frac{1}{\delta+\gamma}\left[-e^{-rt}\mu_{x+t} + {}_tV_x^M\left(\mu_{x+t}+\delta+\gamma\right)\right] \\ &= -\frac{1}{\delta+\gamma}\left[-e^{-rt}\mu_{x+t} + \left(e^{-rt} - (\delta+\gamma){}_tV_x^a\right)\left(\mu_{x+t}+\delta+\gamma\right)\right] \\ &= -e^{-rt} + {}_tV_x^a\left(\mu_{x+t}+\delta+\gamma\right) \end{aligned} \tag{19} $$

where

$$ \begin{aligned} {}_tV_x^M &= \int_0^\infty e^{-rt}e^{-rs}e^{-\delta s}{}_sP_{x+t}\,\mu_{x+t+s}\,ds \\ &= e^{-rt}e^{-rs}e^{-\delta s}\big|_{s=0} + \int_0^\infty -\left(\delta+\gamma\right)e^{-rt}e^{-rs}e^{-\delta s}{}_sP_{x+t}\,ds \\ &= e^{-rt} - (\delta+\gamma)\int_0^\infty e^{-rt}e^{-rs}e^{-\delta s}{}_sP_{x+t}\,ds \\ &= e^{-rt} - (\delta+\gamma){}_tV_x^a \end{aligned} $$

Rearranging the equation (19) yields

$$ {}_tV_x^a = \frac{\frac{d}{dx}{}_tV_x^a + e^{-\gamma t}}{\mu_{x+t}+\delta+\gamma}. \tag{20} $$

The four conditions of the annuity product for equation (19) are as follows:

1. When $\mu_{x+t}+\delta+\gamma > 0$ and $\frac{d}{dx}{}_tV_x^a \le 0$, then ${}_tV_x^a \le \frac{e^{-\gamma t}}{\mu_{x+t}+\delta+\gamma}$.
2. When $\mu_{x+t}+\delta+\gamma < 0$ and $\frac{d}{dx}{}_tV_x^a \le 0$, then ${}_tV_x^a \ge \frac{e^{-\gamma t}}{\mu_{x+t}+\delta+\gamma}$.
3. When $\mu_{x+t}+\delta+\gamma > 0$ and $\frac{d}{dx}{}_tV_x^a \ge 0$, then ${}_tV_x^a \ge \frac{e^{-\gamma t}}{\mu_{x+t}+\delta+\gamma}$.
4. When $\mu_{x+t}+\delta+\gamma < 0$ and $\frac{d}{dx}{}_tV_x^a \ge 0$, then ${}_tV_x^a \le \frac{e^{-\gamma t}}{\mu_{x+t}+\delta+\gamma}$.

The equation of boundary reserve is ${}_tV_x^a = \dfrac{e^{-\gamma t}}{\mu_{x+t} + \delta + \gamma}$ when $\dfrac{d}{dx}\,{}_tV_x^a$ is equal to zero. With positive value of $e^{-\gamma t}$, rearranging the equation of boundary reserve (20) into

$$\left(\mu_{x+t} + \delta + \gamma\right){}_tV_x^a = e^{-\gamma t}, \tag{21}$$

we obtain the only possible feature of hyperbola with center at $(-\delta-\gamma, 0)$ on the $(\mu_{x+t}$, ${}_tV_x^a)$ plane. The two asymptotes of the hyperbola are given by $\mu_{x+t} = -\delta-\gamma$ and ${}_tV_x^a = 0$. The feature with positive value of $e^{-\gamma t}$ lays in the 1st and 3rd quadrants with respect to the center, as shown in Figure 7.

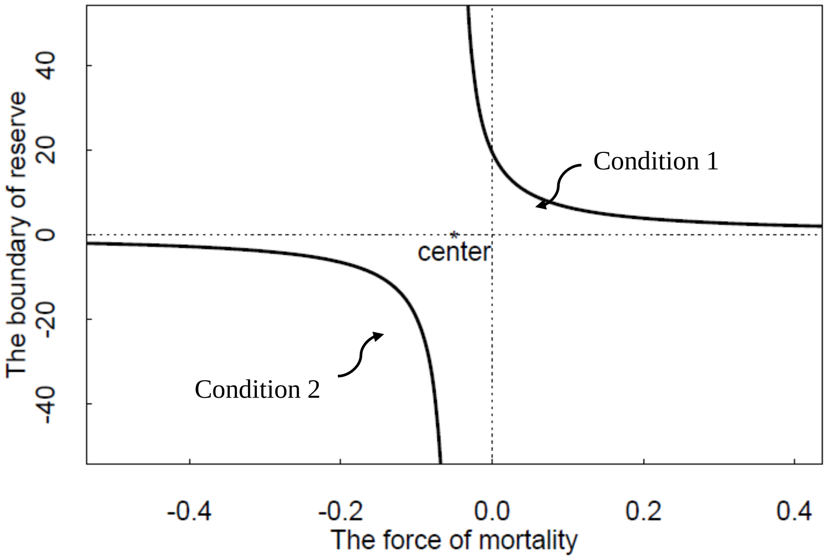

Figure 7 The boundary reserve with positive γ on the $(\mu_{x+t}, {}_tV_x)$ plane

Since the slope $\dfrac{d\,{}_tV_x^a}{d\,\mu_{x+t}}$ of the feature in Figure 7 is negative and if we assume that μ_{x+t} is an increasing function of age x we can get that $\dfrac{d}{dx}\,{}_tV_x^a$ is negative as required criterion in condition 1 and 2. The positive $\dfrac{d}{dx}\,{}_tV_x^a$ as required criterion in condition 3 and 4 is impossible existing in our specified annuity product unless μ_{x+t} is a decreasing function of age x, but this is against nature of aging. We then only focus on the cases when $\frac{d}{dx}\,{}_tV_x^a$ is negative in condition 1 and 2.

Next, we transform the feature on the (μ_{x+t}, ${}_tV_x^a$) plane into that on the ($x+t$, ${}_tV_x^a$) plane by providing μ_{x+t} a given function of age x as shown in Figure 8. The feature in Figure 8 is the case in condition 1 with the criterion of $\mu_{x+t}+\delta + \gamma>0$ on the ($x+t$, ${}_tV_x^a$) plane. This feature displays the upper bound of the expected reserve in condition 1. For the same reason in section 2, the case of condition 2 is not shown on the (μ_{x+t}, ${}_tV_x^a$) plane.

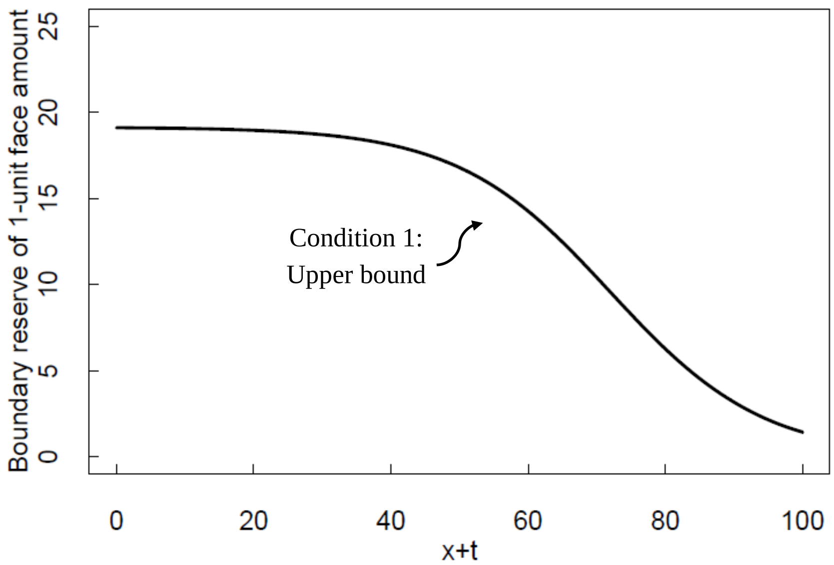

Figure 8 The boundary reserve with positive γ on ($x+t$, ${}_tV_x$) plane

Note that the illustration is using the Makeham model $\mu_{x+t} = 9.566 \times 10^{-4} + 5.162 \times 10^{-5} \times 1.09369^{x+t}$, which is cited from Melnikov and Romaniuk (2006) and the original data is based on the mortality rates from 1959 to 1999 in American (Pollard, 1973).

Unlike the outcome in life insurance, the annuity product with positive γ cannot yield a feature of horizontal line that is confirmation in determining if the optimal mitigation strategy exists. Even though we can only get the secondary strategy for the annuity product in such product structure, the created effect of mortality risk in the annuity product can still mitigate part of the longevity risk within the policy. Thus, given a positive γ we can use the secondary strategy, risk diminishing, to apply on the annuity product to respond the demand of product risk.

4.2 Numerical Illustrations

We assume that the face amount is US$10,000 for the annuity product and the premium is paid by single premium. Assume that the company only sells the specified annuity product with given value of δ. The strategy is to set the value of γ. γ can be any given positive value to lower the total risk of the product. The basic assumption of annuity product as shown in the Table 9

Table 9

The Basic Information of the Illustrated Annuity Product

Age of insured	25, 45
Gender	Male
Face amount	10,000
The initial value of force of interest rate (δ)	4%
Annum payable	10,000 compounded by -$\gamma(t)$
Benefit period	Whole life
Method of paying premium	Single premium

Under the strategy of the annuity products with the force of amount, we can diminish the longevity risk of the annuity products within the policy when mortality rate is changed. We take a whole life annuity as an illustrated case. In Table 10, we show that the longevity risk of the annuity product is less than that with the force of amount.

Table 10

The Liability at the End of the 5th Policy Year of Illustrated Annuity Product for Different Mortality Bases

A: γ (=4%) <δ (=4%)					
	(1)	(2)	(3)	(4)=[(2)-(1)]/(1)	(5)=[(3)-(1)]/(1)
age	Basis	20% Up Shock	20%Down Shock	Reserve Changed	Reserve Changed
25	90,472	89,608	91,389	-0.955%	1.014%
45	79,509	77,469	81,759	-2.565%	2.830%
B: γ (=0%) <δ (=4%)					
25	195,256	191,308	199,719	-2.022%	2.286%
45	152,721	146,562	159,903	-4.032%	4.703%

The product A with δ, 4% and γ, 4% is compare to product B with δ, 4% and γ, 0%. The product A is exposed to both the mortality risk and the longevity risk and the mortality risk is less than the longevity risk. The product B is exposed to longevity risk only. When the mortality rate is changed by 20% shock, the expected reserve of both products is changed. With increase of reserve as indicated in column 5 with respect to 20% down shock of mortality rate, product A is exposed to risk of 1%~3% more than that in basis mortality rate. And the product B is exposed to that of 2~5% more than that in basis mortality rate by the changes of mortality rate. We show that the secondary strategy can be elaborated to diminish the risk within the policy by creating mortality risk in the annuity product to lower the exposure of longevity risk and the strategy helps us to design such products upon the demand of risk exposure.

5. Conclusion

We discover that natural hedging strategy through product design is an important step that we can immunize/mitigate the mortality rate risk within a policy. The key point is that we should take the risk rooting in “when to pay” and the risk rooting in “how much to pay” into consideration in risk mitigation technique. We introduce a factor γ, the force of amount, as a risk factor rooting in “how much to pay” in the product design.

We utilize the γ to design a death benefit protection life insurance that the mortality rate risk is immunized. We deduce the optimal natural hedging strategy within a policy with the criteria of settings on γ as to let $\gamma = \delta$. When the γ is equal to δ, the specified product appears no risk with the changes of mortality rate. The mortality rate risk is immunized in such kinds of product with death benefit protection in all ages. When the γ is not equal to δ, the strategy is used to diminish the risk rather than to immunize the risk. If $\delta > \gamma$, the death benefit protection product is exposed to mortality risk rather than longevity risk with the changes of mortality rate and the life insurance product with settings of γ is less exposed to mortality risk than the one without settings of γ.

In the case of the death benefit life insurance product, the γ factor is creating a way to mitigate the mortality risk of life insurance. And in the case of annuity product, the γ factor is used to mitigate the longevity risk of the annuity products. Based on our theoretical analysis, the annuity products can only deduce a secondary strategy to diminish risk within the policy by creating a mortality risk factor γ.

Following the optimal strategy in the product design, the insurance product has a lot more possibility in engaging to financial instrument without considering the mortality rate risk. Our finding can also provide a further research or re-design of previous studies that ignore the existence of a mortality rate risk in an insurance product valuation or risk management consideration.

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