Catastrophe Aversion and Risk Equity under Dependent Risks

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Abstract

In this paper, we compare risky social situations that differ by the degree of dependency among individual risks. We show that more correlation between the risks faced by two individuals always induces a more catastrophic distribution of fatalities. We derive the necessary and sufficient condition so that an equity-increasing risk transfer between two individuals always implies a more catastrophic distribution. This condition holds when a change in risk equity does not decrease the correlation between the two risks, or when one individual's death (or survival) is certain. We generalize some of the results in a N-agent world by assuming that the risk transfer between any two agents does not affect the dependence structure among the remaining $N-2$ agents.

Key words: risk equity, catastrophe aversion, correlation, dependence.

1 Introduction

Perhaps the most common measure to assess and manage social risks is the expected number of fatalities (or injuries). However, the expectation does not account for important dimensions of risk. It neither reflects society's preferences to avoid large scale accidents, nor does it capture concerns over inequalities in the distribution of risk across individuals. Alternative criteria for managing public risks have therefore emerged. These criteria aim at limiting the maximum probable loss or the maximum individual risk. In his seminal work, Ralph Keeney (1980) formally defined the concepts of risk equity and catastrophe aversion. Risk equity is an *ex ante* concept corresponding to a preference for equalizing the probability of dying across agents. Catastrophe aversion is an *ex post* concept, which corresponds to a preference for a mean-preserving concentration of the distribution of fatalities (Adler et al. 2014).

Assuming independent risks, Keeney (1980) showed that the two concepts are always in conflict. Whenever one increases risk equity, the distribution of fatalities becomes more catastrophic and vice versa. This result is challenging because it illustrates the tension between two reasonable objectives of risk managers: limiting the risk burden to individuals and to society as a whole. It has received a lot of attention in the operations research and management literature (e.g., Fishburn 1984; Keeney and Winkler 1985; Sarin 1985; Fishburn and Straffin 1989; Fishburn and Sarin 1991, 1994, 1997; Gajdos et al. 2010), and more recently in the economics and social choice literature (Bommier and Zuber 2008; Fleurbaey 2010; Fleurbaey and Bovens 2012; Adler et al. 2014).

In this paper, we examine an aspect of the problem that has so far been largely overlooked—the dependence structure of social risks. Arguably, interdependent risks are the rule rather than the exception. This is particularly true for potentially catastrophic risks (e.g., storms, terrorist attacks, climate change or industrial accidents). We start from a two-agent world and prove that the more correlated the risks are, the more catastrophic is the distribution of fatalities. We then derive the necessary and sufficient condition under which an equity-increasing risk transfer implies a more catastrophic distribution. This condition pins down the effects of more risk equity on both the marginal distributions of the two risks and their correlation. We show in particular that this condition holds when a change in risk equity does reduce the correlation between the two risks, or when one individual's death (or survival) is certain.

Extending the analysis to more than two agents is challenging. Pairwise correlations provide an insufficient statistic to map out the dependence structure of multiple risks (Embrechts et al. 2002). Nonetheless, we are able to generalize some of the results obtained for the two-agent case by imposing that, in a N-agent world, risk transfers between any two agents do not affect the dependence structure of the remaining $N-2$ agents, and by restricting the notion of more catastrophic.

2 Catastrophic and correlated risks

2.1 Definitions and notations

Consider a population of N agents ($i = 1, ..., N$), each of whom faces an individual probability of dying $p_i \in [0,1]$. The risk of death is modeled as a Bernoulli random variable $\tilde{x}_i$, which takes the value 1 (i.e., agent i dies) with probability p_i and 0 otherwise. We are interested in the distribution of fatalities:

$$\tilde{d} := \sum_{i=1}^{N} \tilde{x}_i.$$

As in Adler et al. (2014), we define a more catastrophic distribution based on the definition of second-order stochastic dominance (Rothschild and Stiglitz 1970).

Definition 1. *A distribution of fatalities $\tilde{d}$ is more catastrophic than a distribution $\tilde{d}'$ if for any concave function f, $\mathsf{E}[f(\tilde{d})] \leqslant \mathsf{E}[f(\tilde{d}')]$.*

Observe that when f is linear, $\mathsf{E}[\tilde{d}] = \mathsf{E}[\tilde{d}']$. This implies that $\tilde{d}$ is a mean-preserving spread of $\tilde{d}'$. Keeney's (1980, Theorem 2 p. 532) definition of "catastrophe avoidance" considers a particular case of definition 1, in which $\tilde{d}$ and $\tilde{d}'$ apply to binary risks with one safe outcome (i.e., an outcome that implies no fatality at all). Keeney's result means that a social planner who behaves consistent with expected utility toward the number of fatalities is catastrophe averse iff the (social) vNM utility function f is concave.[1] Now, let $f(d) = -d^2$. It is immediate that a more catastrophic distribution must have a greater variance, but not the other way around. This leads to our second definition.

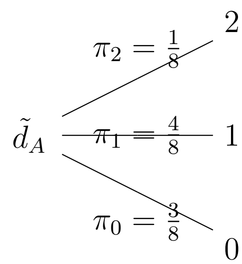
Definition 2. *A distribution of fatalities $\tilde{d}$ is more variable than a distribution $\tilde{d}'$ iff $\mathsf{var}(\tilde{d}) \geqslant \mathsf{var}(\tilde{d}')$.*

To begin with, we focus on two agents whose risks of death $\tilde{x}_1$ and $\tilde{x}_2$ may depend on each other. We introduce the probability space Ω comprised of a finite (but potentially large) number of states S to describe the interaction between the two risks. Let us for now assume that Ω has $S := 8$ equiprobable states $(\omega_1, ..., \omega_8)$ and consider two agents, 1 and 2, who face the probability of dying p_1 and p_2,

[1]Note however that it is not obvious that societal preferences should always display catastrophe aversion. For instance, Schelling (1968) in his seminal paper on the value of life presents an example supporting catastrophe prone preferences in the context of a family risk.

respectively.[2] Throughout the paper, we will make use of the following matrix notation to illustrate risky social situations.

$$\begin{array}{c} \\ \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \\ \omega_5 \\ \omega_6 \\ \omega_7 \\ \omega_8 \end{array} \begin{array}{c} \begin{array}{cc} \tilde{x}_1 & \tilde{x}_2 \end{array} \\ \left[\begin{array}{cc} 1 & 1 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{array} \right] \end{array}$$



(A)

Each of the $S := 8$ rows in this matrix corresponds to one possible state of the world. For $s = 1, ..., 8$, the ith value in row s can be interpreted as $\tilde{x}_i(\omega_s)$, which is the value taken by the random variable $\tilde{x}_i$ in state ω_s. In situation (A), there is only one state in which the two agents die simultaneously (ω_1). Each state in situation (A) has the same probability $1/S = \frac{1}{8}$ to occur. $\tilde{x}_1(\omega)$ can take two values: 1 with probability $p_1 := \Pr(\tilde{x}_1 = 1) = \frac{1}{2}$ (corresponding to the occurrence of a state in which the first agent dies) and 0 with probability $1 - p_1 = \frac{1}{2}$ (corresponding to a state in which the first agent survives). Likewise, the probability that the second agent dies is $p_2 := \Pr(\tilde{x}_2 = 1) = \frac{1}{4}$. Note that p_2 does not depend on the realizations of $\tilde{x}_1$ and *vice versa*: $\Pr(\tilde{x}_2 = 1|\tilde{x}_1 = 1) = \Pr(\tilde{x}_2 = 1)$ and $\Pr(\tilde{x}_1 = 1|\tilde{x}_2 = 1) = \Pr(\tilde{x}_1 = 1)$. In other words, the two risks are independent.

Based on the matrix notation the computation of the distribution of fatalities $\tilde{d}_A := \tilde{x}_1 + \tilde{x}_2$ becomes straightforward. We only need to sum the values in each row to find that $\pi_0 := \Pr(\tilde{d} = 0) = \frac{3}{8}$, $\pi_1 := \Pr(\tilde{d} = 1) = \frac{4}{8}$, and $\pi_2 := \Pr(\tilde{d} = 2) = \frac{1}{8}$. The corresponding distribution of fatalities is represented by the probability tree next to the matrix.

2.2 Correlation and the distribution of fatalities

In the two-agent world there is a fundamental relationship between the distribution of fatalities and the correlation of the risks. The following proposition assumes no change in the marginal distributions (i.e., the parameters p_1 and p_2

[2]In our examples, we use equiprobable states to ease the exposition. We emphasize, however, that the results also hold for unequal state probabilities because each state can be broken down into several equiprobable states.

are kept fixed). Thus, at this stage, there is no change in risk equity involved.

Proposition 1. *Under $N = 2$, the four following statements are equivalent:*

(i) the probability of simultaneous fatalities increases,

(ii) the correlation between the individual risks increases,

(iii) the distribution of fatalities is more catastrophic (Definition 1),

(iv) the distribution of fatalities is more variable (Definition 2).

Proof. We first prove that the distribution becomes more catastrophic iff the probability of simultaneous fatalities, $\pi_2 := \Pr(\tilde{x}_1 = 1, \tilde{x}_2 = 1)$, increases. For $N = 2$ agents, $\mathsf{E}[f(\tilde{d})] = \pi_0 f(0) + \pi_1 f(1) + \pi_2 f(2)$, where $\pi_i := \Pr(\tilde{d} = i)$ for $i = 0, 1, 2$. We know that $\mathsf{E}[\tilde{d}] = p_1 + p_2$, so that $\pi_1 + 2\pi_2 = p_1 + p_2$ and $\pi_0 + \pi_1 + \pi_2 = 1$. Using these two equalities, we can express π_0 and π_1 as functions of π_2: $\mathsf{E}[f(\tilde{d})] = (1 - (p_1 + p_2 - 2\pi_2) - \pi_2)f(0) + (p_1 + p_2 - 2\pi_2)f(1) + \pi_2 f(2)$. This expression can be further simplified to

$$\mathsf{E}[f(\tilde{d})] = (1 - p_1 - p_2)f(0) + (p_1 + p_2)f(1) + \pi_2(f(0) - 2f(1) + f(2)).$$

By Jensen's inequality we have $f(0) - 2f(1) + f(2) \leqslant 0$ for all f concave. Therefore, $\mathsf{E}[f(\tilde{d})]$ decreases iff π_2 increases. Thus (iii)$\Leftrightarrow$(i).

Next, we turn to the joint probability of two random Bernoulli variables. We have

$$\pi_2 = \Pr(\tilde{x}_1 = 1, \tilde{x}_2 = 1) = \mathsf{E}[\tilde{x}_1\tilde{x}_2] = p_1 p_2 + \rho\sqrt{p_1(1 - p_1)}\sqrt{p_2(1 - p_2)}$$

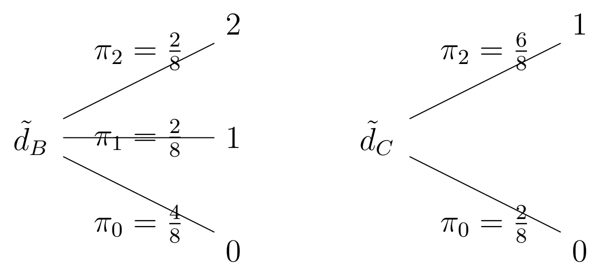
by the definition of the correlation ρ coefficient between the two risks $\tilde{x}_1$ and $\tilde{x}_2$. It is easy to see that π_2 increases whenever ρ increases. Thus (i)$\Leftrightarrow$(ii). Finally, observe that

$$\mathsf{var}(\widetilde{x}_1 + \widetilde{x}_2) = p_1(1 - p_1) + p_2(1 - p_2) + 2\rho\sqrt{p_1(1 - p_1)}\sqrt{p_2(1 - p_2)},$$

which increases iff correlation ρ increases. Thus (ii)$\Leftrightarrow$(iv). ■

Let us give an intuition for this result by modifying the introductory example (A). We show that the distribution of fatalities can become more or less catastrophic, depending only on the interaction between the two individual risks of death. In situation (B) below we alter the dependence structure between $\tilde{x}_1$ and $\tilde{x}_2$ such that the occurrence of two simultaneous fatalities becomes most likely. In contrast, situation (C) illustrates a dependence structure under which two simultaneous fatalities are impossible:

$$
\begin{array}{c}
\begin{array}{cc} \tilde{x}_1 & \tilde{x}_2 \end{array} \\
\left[\begin{array}{cc}
1 & 1 \\
1 & 1 \\
1 & 0 \\
1 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0
\end{array}\right] \\
(B)
\end{array}
\qquad\qquad
\begin{array}{c}
\begin{array}{cc} \tilde{x}_1 & \tilde{x}_2 \end{array} \\
\left[\begin{array}{cc}
1 & 0 \\
1 & 0 \\
1 & 0 \\
1 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 1 \\
0 & 1
\end{array}\right] \\
(C)
\end{array}
$$



The risky situation depicted in (B) leads to the most catastrophic distribution of fatalities. This situation in which the worst outcome becomes most likely is called the *comonotonic* dependence structure. By contrast, the risky situation depicted in (C) gives rise to the least catastrophic distribution of fatalities. This situation is also called the *anticomonotonic* dependence structure because fatal outcomes are ordered in reverse order, which minimizes the probability to observe simultaneous fatalities.

The result in Proposition 1 is linked to Epstein and Tanny (1980)'s concept of "generalized correlation." This concept corresponds to a general statistical definition for the statement that the random variables $\tilde{x}_1$ and $\tilde{x}_2$ are more correlated than the random variables $\tilde{x}'_1$ and $\tilde{x}'_2$ (see also Tchen 1980 and Wright 1987). Epstein and Tanny show that this statistical definition is equivalent to $Eu(\tilde{x}_1, \tilde{x}_2) \leqslant Eu(\tilde{x}'_1, \tilde{x}'_2)$ whenever the cross partial derivatives $u_{12} \leqslant 0$. Now take $u(x_1, x_2) = f(x_1 + x_2)$ so that $u_{12} \leqslant 0$ is equivalent to $f'' \leqslant 0$. This shows that an increasing "generalized correlation" between $\tilde{x}_1$ and $\tilde{x}_2$ is sufficient for obtaining a more catastrophic distribution $\tilde{x}_1 + \tilde{x}_2$. Proposition 1 shows that more correlation is necessary and sufficient to obtain a more catastrophic distribution $\tilde{x}_1 + \tilde{x}_2$ when $\tilde{x}_1$ and $\tilde{x}_2$ are Bernouilli random variables.

We know from Proposition 1 that the distribution of fatalities is the most (least) catastrophic when the correlation between the risks of the two agents is maximized (minimized). In the case of two agents, it is always possible to fully identify the least and most catastrophic distribution of fatalities. This result is summarized in the following lemma, and will turn out to be useful later. In particular, we observe that the range of correlation between two risks depends on their marginal probabilities.

Lemma 1. *The correlation ρ between two Bernoulli random variables $\tilde{x}_1$ and $\tilde{x}_2$ with $p_1 > p_2$ lies in the following interval:*

- $\rho \in \left[-\frac{\sqrt{p_1 p_2}}{\sqrt{1-p_1}\sqrt{1-p_2}}, \frac{\sqrt{p_2}\sqrt{1-p_1}}{\sqrt{p_1}\sqrt{1-p_2}}\right]$, *if* $p_1 + p_2 \leqslant 1$
- $\rho \in \left[-\frac{\sqrt{1-p_1}\sqrt{1-p_2}}{\sqrt{p_1 p_2}}, \frac{\sqrt{p_2}\sqrt{1-p_1}}{\sqrt{p_1}\sqrt{1-p_2}}\right]$, *if* $p_1 + p_2 > 1$
- $\rho = 0$, *if* $p_i = 0$ *or* $p_i = 1$ *for* $i = 1, 2$
- $\rho = 1$, *if* $p_i = 0$ *or* $p_i = 1$ *for* $i = 1, 2$

Proof. The original proof of Lemma 1 is given in Tchen (1980) and in Meilijson and Nadas (1979). Here, we only provide a sketch of the idea behind the proof. We start with the minimum correlation, which is attained when the Bernoulli variables are countermonotonic, i.e. when the number of simultaneous deaths is minimized. We know that $\pi_2 = \mathsf{E}[\tilde{x}_1\tilde{x}_2] = \Pr(\tilde{x}_1 = 1, \tilde{x}_2 = 1) = \max(0, p_1 + p_2 - 1)$. There are two cases $p_1 + p_2 \leqslant 1$ and $p_1 + p_2 > 1$. In both cases, the minimum correlation is equal to

$$\frac{\max(0, p_1 + p_2 - 1) - p_1 p_2}{\sqrt{p_1 (1 - p_1) p_2 (1 - p_2)}}.$$

By considering these two cases separately and after simplifications, one obtains the expressions of the minimum correlation presented in the proposition.

The maximum correlation is obtained when the Bernoulli variables are comonotonic, i.e. when the number of simultaneous fatalities is maximized. For the two-agent world this means that in each state in which agent 2 dies, agent 1 dies as well. Because the maximum probability of simultaneous fatalities is equal to $\Pr(\tilde{x}_1 = 1, \tilde{x}_2 = 1) = \min(p_1, p_2) = p_2$, their maximum correlation equals

$$\frac{p_2 - p_1 p_2}{\sqrt{p_1 (1 - p_1) p_2 (1 - p_2)}} = \frac{\sqrt{p_2}\sqrt{1 - p_1}}{\sqrt{p_1}\sqrt{1 - p_2}}.$$

One practical remark on Lemma 1 seems in order. The natural bounds of the correlation interval are only obtained in the special case where $p_1 = p_2 = \frac{1}{2}$. This implies that for most binary risks, the interval of attainable correlation is narrower than $\rho \in [-1, 1]$. ■

3 Risk equity and its implications

3.1 Keeney's (1980) result

Following Keeney (1980) and the subsequent literature cited in the introduction, we define risk equity using the concept of a "Pigou-Dalton" transfer in risk. This is a non-leaky transfer of probability mass from a more exposed to a less exposed individual, so that the transfer does not reverse the ranking of the two individuals in terms of their probability to die.[3]

Definition 3. *A distribution of fatalities $\tilde{d}$ is more equitable than a distribution $\tilde{d}'$ iff it is obtained by a Pigou-Dalton transfer δ from a more exposed agent j to a less exposed agent i would reduce the exposure of j and raise the exposure of i without switching their ranking in terms of absolute exposure and without changing other individuals' exposure. I.e., the probabilities of dying* before *the transfer are p_i and p_j, with $p_j > p_i$;* after *the transfer, the probabilities of dying are $p'_i = p_i + \delta$ and $p'_j = p_j - \delta$, where $\delta \leqslant \frac{p_j - p_i}{2}$.*

A Pigou-Dalton risk transfer thus decreases the "gap" in terms of risk exposure of agents i and j (Rothschild and Stiglitz 1973; Adler et al. 2014). Proposition 2 asserts that, under the assumption of independent risks, more risk equity implies a more catastrophic distribution of fatalities.[4]

Proposition 2. *(Keeney 1980) Assume that the risks to N agents are independent. Then, any Pigou-Dalton transfer in risk between two agents leads to a more catastrophic distribution of fatalities if the risks to the $N - 2$ agents are still independent after the Pigou-Dalton transfer.*

Proof. Based on the equivalence between an increase in catastrophic risk and in variance (Proposition 1), it is easy to prove Proposition 2. Indeed, a Pigou-Dalton transfer in risk increases the variance of the distribution of fatalities as the change in the variance is equal to $\mathsf{var}[\tilde{d}'] - \mathsf{var}[\tilde{d}] = [(p_i + \delta)(1 - p_i - \delta) + (p_j - \delta)(1 - p_j + \delta)] - [p_i(1 - p_i) + p_j(1 - p_j)]$, which is positive for any $\delta \leqslant \frac{p_j - p_i}{2}$. ■

Our next objective is to extend this result to the case of two dependent risks. To illustrate the complexity arising from the interaction between the two risks,

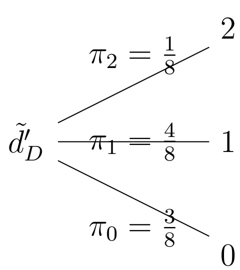
[3]Interestingly, Cox (2012) argues that—unlike income—mortality risk is not fungible and cannot be transferred from one individual to another. We note however, that the definition need not imply a physical transfer of risk from individual j to individual i. It merely implies the existence of different policy options that differ in how risk is shared among i and j.

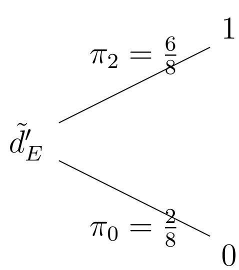
[4]Note that this result is in fact a slight generalization of Keeney (1980). Indeed, as said above, Keeney considered a particular case of the definition of a more catastrophic distribution that we consider here.

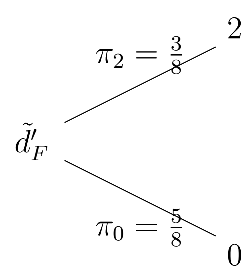
we set out from example (A) and present three feasible risk transfers and their respective effect on the distribution of fatalities. Remember that in situation (A) the risks of the two agents were independent. Here and in contrast to Keeney (Proposition 2), we do not presume that the risks remain independent after the Pigou-Dalton transfer.

Consider the following situations labeled (D), (E), and (F), respectively. Risks after a Pigou-Dalton transfer of $\delta = \frac{1}{8}$ are denoted by $\tilde{x}_1'$ and $\tilde{x}_2'$. After the transfer, both agents face the same probability to die ($p_1' := p_1 - \delta = p_2' := p_2 + \delta = \frac{3}{8}$) in all three situations (D), (E), and (F). However, in each of the situations the two risks are no longer independent. Consequently, Proposition 2 does not apply any longer.

$$
\begin{array}{ccc}
\begin{array}{cc} \tilde{x}_1' & \tilde{x}_2' \end{array} & \begin{array}{cc} \tilde{x}_1' & \tilde{x}_2' \end{array} & \begin{array}{cc} \tilde{x}_1' & \tilde{x}_2' \end{array} \\
\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} &
\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} &
\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \\
(D) & (E) & (F)
\end{array}
$$







The new correlation ρ' between the agents' risks can be computed as follows:

$$\rho' := \mathsf{corr}(\tilde{x}_1', \tilde{x}_2') = \frac{\mathsf{E}[\tilde{x}_1'\tilde{x}_2'] - (p_1 - \delta)(p_2 + \delta)}{\sqrt{(p_1 - \delta)(1 - p_1 + \delta)}\sqrt{(p_2 + \delta)(1 - p_2 - \delta)}}.$$

For the above situations, we have: $\rho_D' \approx -0.06$, $\rho_E' = -0.6$, and $\rho_F' = 1$, respectively. The distribution of fatalities $\tilde{d}' := \tilde{x}_1' + \tilde{x}_2'$ after each of the feasible Pigou-Dalton transfers is fully characterized by the corresponding probability trees. Compared to situation (A), the distribution of fatalities becomes strictly more catastrophic in situation (F), strictly less catastrophic in situation (E), and is identical in situation (D).

Situations $(A)-(F)$ make it clear that the change in the distribution of fatalities is governed by two sources: (i) the effect of the Pigou-Dalton risk transfer on the marginal distributions, and (ii) the change in correlation induced by the transfer. Both changes have an impact on how catastrophic the distribution of fatalities is. Situations $(D)-(F)$ have illustrated that an increase (or decrease) in catastrophic risk might be due to a change in correlation only. This result underlines that it is not straightforward to extend Proposition 2, and there is no hope to generically sign the comparative statics analysis without further restrictions on the correlation.[5]

3.2 The necessary and sufficient condition

In this section we identify the necessary and sufficient condition under which a Pigou-Dalton transfer in risk leads to a more catastrophic distribution of fatalities. This condition includes the special case in which the risks have identical correlation before and after the transfer. Nevertheless, we provide a more general analysis in which we allow the correlation to change before and after the transfer, as in the examples above. We believe that this degree of generality is important for some real world applications.[6] We show that a Pigou-Dalton transfer makes the distribution of fatalities more catastrophic whenever the correlation after the transfer is larger than a specific threshold.

Proposition 3. *Assume $N=2$ and $p_1 > p_2$. Let ρ denote the correlation between the initial risks $\tilde{x}_1$ and $\tilde{x}_2$. After a Pigou-Dalton transfer $\delta \in \left[0, \frac{p_1-p_2}{2}\right]$, the distribution of fatalities becomes more catastrophic iff the correlation ρ' between $\tilde{x}'_1$ and $\tilde{x}'_2$ is larger than the critical level of correlation ρ^*, i.e.*

$$\rho' \geqslant \rho^* := \frac{\delta(p_2 - p_1 + \delta) + \rho\sqrt{p_1}\sqrt{1-p_1}\sqrt{p_2}\sqrt{1-p_2}}{\sqrt{1-p_1+\delta}\sqrt{p_1-\delta}\sqrt{p_2+\delta}\sqrt{1-p_2-\delta}}. \tag{1}$$

Proof. We know from the proof of Proposition 1 that the distribution of fatalities

[5]For $S := 8$ states $(\omega_1, ..., \omega_8)$, it is impossible to find a situation in which the correlation between $\tilde{x}'_1$ and $\tilde{x}'_2$ is equal to 0 (i.e., in which the risks are still independent *after* the risk transfer) and for which the result of Keeney (Proposition 2) would hence hold. It is, however, possible to construct such a situation by invoking more states of the world. We provide one such example in Appendix A.1.

[6]Consider for instance the introduction of a new technology, like a better navigation system in cars. This technology will tend to reduce individual differences in exposure to the risk of car accident (due to, e.g., driving abilities, or car quality). Therefore this technology will tend to increase risk equity in the society.Yet, it may also increase the dependence among individual risks, because of the risk of a general failure of the navigation system. Hence, this new technology may increase both risk equity and risk dependency in the society.

becomes more catastrophic iff $\pi_2 = \mathsf{E}[\tilde{x}_1\tilde{x}_2]$ increases. To grasp the effect of the risk transfer on the distribution of fatalities, we compute $\mathsf{E}[\tilde{x}_1\tilde{x}_2]$ (before the transfer) and $\mathsf{E}[\tilde{x}_1'\tilde{x}_2']$ (after the transfer). We find that

$$\pi_2 = \mathsf{E}[\tilde{x}_1\tilde{x}_2] = p_1p_2 + \rho\sqrt{p_1(1-p_1)}\sqrt{p_2(1-p_2)}, \tag{2}$$

and

$$\begin{aligned}\pi_2' &= \mathsf{E}[\tilde{x}_1'\tilde{x}_2'] \\ &= (p_1-\delta)(p_2+\delta) + \rho'\sqrt{(p_1-\delta)(1-p_1+\delta)}\sqrt{(p_2+\delta)(1-p_2-\delta)}. \end{aligned} \tag{3}$$

Thus, $\pi_2' \geqslant \pi_2$ iff the correlation ρ' is sufficiently high (i.e., iff condition (1) is satisfied). ■

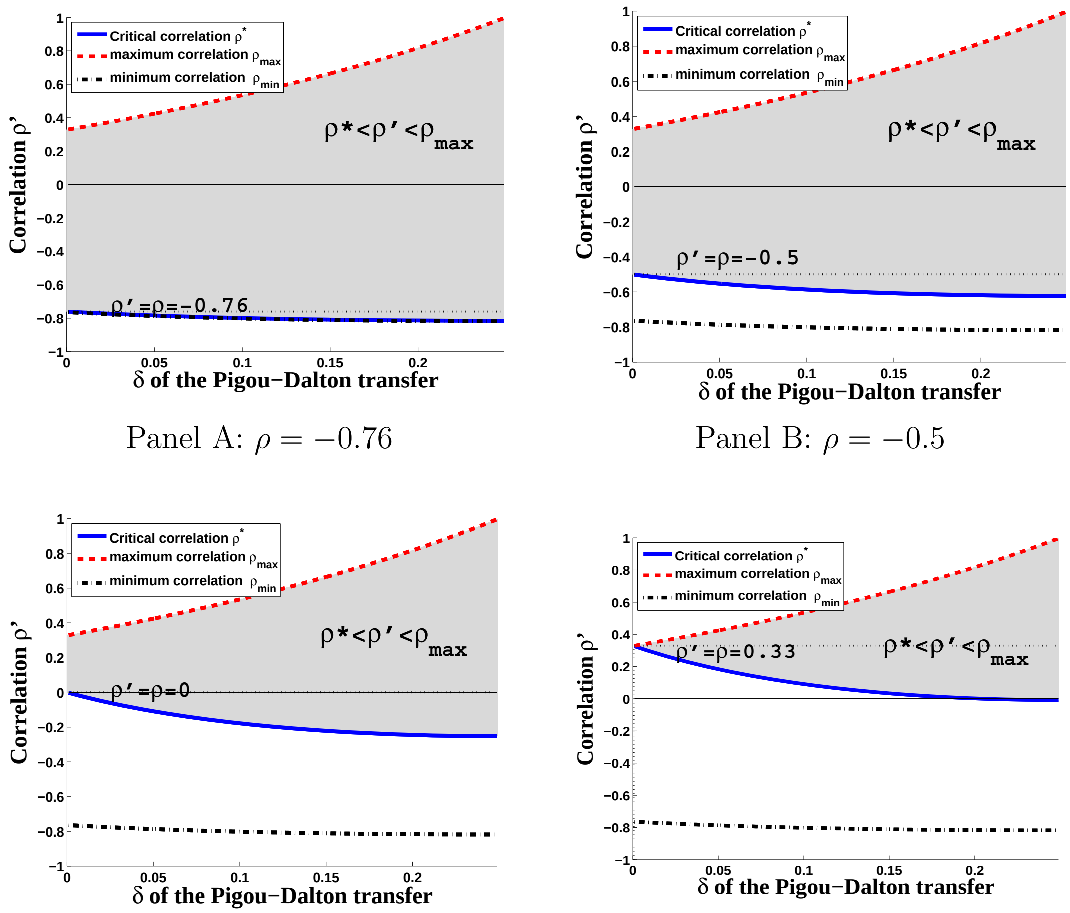
As demonstrated, a Pigou-Dalton transfer may have two distinct effects on the distribution of fatalities; one due to the change in the marginal distributions and the other one due to a (possible) change in the correlation between the two risks. Condition (1) in Proposition 3 depends on both effects and we therefore seek to disentangle them.

The effect on the marginal distributions corresponds to a change in δ, keeping the correlation structure fixed: $\rho' = \rho$. The effect on dependency corresponds to a change in correlation from ρ to ρ', assuming no changes in the marginal distributions (i.e., $\delta = 0$). However, it is fallacious to consider the two effects separately as the change in correlation is bounded (see Lemma 1), and the bounds depend on δ. Figure 1 displays the comparative statics analysis for a numeric example where the two effects are simultaneously at play (as could be expected in real world situations). The grey-shaded areas in the four panels of Figure 1 represent the joint range that the transfer δ (on the x-axis) and the correlation ρ' (on the y-axis) can take on such that the distribution of fatalities is more catastrophic after the Pigou-Dalton transfer (the example assumes that $p_1 = 0.8$, $p_2 = 0.3$, $\delta \in \left[0, \frac{p_1-p_2}{2}\right]$, and analyzes different initial values of ρ).

Some remarks on the comparative statics analysis seem warranted. By Lemma 1, we know that the range over which correlation is defined depends on the marginal distributions. In other words, the range of attainable correlation between $\tilde{x}_1'$ and $\tilde{x}_2'$ is a function of δ because the correlation ρ' between two risks with respective probabilities $p_1 - \delta$ and $p_2 + \delta$ cannot be larger than

$$\rho_{\max}(\delta) := \frac{\sqrt{p_2+\delta}}{\sqrt{p_1-\delta}}\frac{\sqrt{1-p_1+\delta}}{\sqrt{1-p_2-\delta}} \tag{4}$$

(see Lemma 1). The minimum correlation depends on whether $(p_1-\delta)+(p_2+\delta) = p_1 + p_2$ is larger than 1 or not and is also defined by Lemma 1. $\rho_{\max}(0)$ equals the maximum correlation ρ between the initial risks if $\delta = 0$.



Panel A: $\rho = -0.76$ Panel B: $\rho = -0.5$

Panel C: $\rho = 0$ Panel D: $\rho = 0.33$

Figure 1: Correlation domains of ρ' for four initial values of ρ, assuming $p_1 = 0.8$, $p_2 = 0.3$, and $\delta \in \left[0, \frac{p_1 - p_2}{2}\right]$.

Figure 1 illustrates the comparative statics analysis for four situations in which the initial correlation between the two agents' risks is equal to $\rho = \rho_{\min}(0) = -0.76$ (Panel A), $\rho = -0.5$ (Panel B), $\rho = 0$ (Panel C), and $\rho = \rho_{\max}(0) = 0.33$ (Panel D). Each panel plots the critical level of correlation ρ^* as well as the attainable minimum and maximum correlation (4) as a function of δ using the relationships of Lemma 1. Several special cases of Proposition 3 are worth to be discussed in more detail. We summarize them in Proposition 4.

Proposition 4. *Assume $N = 2$ and $p_1 > p_2$. The distribution of fatalities becomes more catastrophic in the following special cases:*

(i) $\rho' \geqslant \rho$ and $\delta > 0$ (which includes Keeney's (1980) result $\rho' = \rho = 0$, and fixed correlation $\rho' = \rho$, as special cases).

(ii) $\delta = 0$ and $\rho' > \rho$ (no change in the marginal distribution of fatalities, but

an increase in correlation).

(iii) *ρ is equal to the minimum correlation (as computed in Lemma 1) between the two risks before the Pigou-Dalton transfer.*

(iv) *ρ' is equal to the maximum correlation (as computed in Lemma 1) between the two risks after the Pigou-Dalton transfer.*

(v) *$p_1 = 1$ (agent 1 is certain to die) or $p_2 = 0$ (agent 2 is certain to survive).*

Proof. We start from the expressions of π_2 and π_2' given by Eqs. (2) - (3) and show that $\pi_2' \geqslant \pi_2$ in each of the above statements (i)-(v).

In order to prove (i), note that by definition a Pigou-Dalton transfer imposes $\delta \leqslant \frac{p_1 - p_2}{2}$, so that $p_1 p_2 \leqslant (p_1 - \delta)(p_2 + \delta)$ and $\sqrt{1 - p_1 + \delta}\sqrt{p_1 - \delta}\sqrt{p_2 + \delta}\sqrt{1 - p_2 - \delta} > \sqrt{p_1}\sqrt{1 - p_1}\sqrt{p_2}\sqrt{1 - p_2}$. For $\rho' \geqslant \rho \geqslant 0$, the proof of (i) follows directly from the expressions of π_2 and π_2' in Eqs. (2) - (3). For $\rho \leqslant \rho' < 0$, the result still holds but the proof is longer and therefore relegated to Appendix A.2.

The proof of (ii) follows directly from Proposition 1. This case isolates the effect of correlation.

In order to prove (iii), observe that there are two cases for which the correlation is minimum (see Lemma 1). If $p_1 + p_2 \leqslant 1$, then $\pi_2 = 0$ obtains after replacing the correlation in Eq. (2) by $\frac{-\sqrt{p_1 p_2}}{\sqrt{1-p_1}\sqrt{1-p_2}}$. Thus, $\pi_2' \geqslant 0 = \pi_2$. If $p_1 + p_2 > 1$, then the expression of π_2 can be simplified to $p_1 + p_2 - 1$. We apply Lemma 1 to the marginals $p_1 - \delta$ and $p_2 + \delta$ and obtain $\pi_2' \geqslant (p_1 - \delta)(p_2 + \delta) + \rho_{\min}\sqrt{1 - p_1}\sqrt{1 - p_2}$. Denoting the minimum level of correlation by $\rho_{\min}$, allows us to simplify the expression to $\pi_2' \geqslant p_1 + p_2 - 1 = \pi_2$.

The proof of (iv) is almost identical. After replacing ρ' in Eq. (2) by the maximum correlation, we find that $\pi_2' = p_2 + \delta$. However $\pi_2 = p_1 p_2 + \rho\sqrt{1 - p_1}\sqrt{1 - p_2} \leqslant p_1 p_2 + \rho_{\max}\sqrt{1 - p_1}\sqrt{1 - p_2}$. Some simplifications yield $\pi_2 \leqslant p_2 < p_2 + \delta = \pi_2'$.

Finally, we prove (v) as follows. First, observe that for $p_1 = 1$ we have $\rho^* := \frac{\delta(p_2 - 1 + \delta)}{\sqrt{\delta(1-\delta)}\sqrt{(p_2+\delta)(1-p_2-\delta)}}$, which equals the minimum bound identified by Lemma 1 (for $p_1 + p_2 > 1$). Second, observe that for $p_2 = 0$ $\rho^* = \frac{\delta(-p_1+\delta)}{\sqrt{(1-p_1+\delta)(p_1-\delta)}\sqrt{\delta(1-\delta)}}$, which equals the minimal bound identified by Lemma 1 (for $p_1 + p_2 \leqslant 1$). Thus, we always obtain $\rho' \geqslant \rho^*$ for these two special cases. ■

Statement (i) of Proposition 4 is apparent in all of the four panels in Figure 1: the horizontal line representing the level $\rho' = \rho$ always belongs to the grey-shaded area where the distribution of fatalities becomes more catastrophic. Statement (ii) follows immediatly from the inspection of the ρ-values at $\delta = 0$. Statement (iii) corresponds to Panel A in Figure 1 for which the correlation between the initial risks is minimum. Statement (iv) is again apparent in all four panels as the dashed line representing the maximum correlation for ρ' always belongs to the grey-shaded area, for which the distribution of fatalities becomes more catastrophic. Statement (v) is a special case of (iii) in which the correlation between the initial risks is equal to 0, which is also the minimum attainable correlation.

4 More than two agents

We already observed in the two-agent world that the distribution of fatalities can become more or less catastrophic when either the correlation or the marginal distributions are altered by a Pigou-Dalton transfer in risk. In this section, we parallel the previous discussion for $N > 2$ agents. As in §2.2, we first discuss whether the distribution of fatalities becomes more (or less) catastrophic when only the dependence structure changes. We then look at the effect of a change in marginal distributions of two risks in the presence of $N > 2$ agents. We demonstrate that, even when the unaffected risks are uncorrelated, it is generally not possible to conclude whether or not the distribution of fatalities becomes more catastrophic. As is well known, pairwise correlation alone does not provide sufficient information to pin down the dependence structure of the $\tilde{x}_1, ..., \tilde{x}_N$ risks. We are, however, able to show that the distribution of fatalities is more variable after a Pigou-Dalton transfer with uncorrelated risks. At the end of §4, we derive the conditions under which the distribution of fatalities becomes more variable after a Pigou-Dalton transfer in risk. In particular, we generalize some of the results in a N-agent world by assuming that the risk transfer between any two agents does not affect the dependence structure among the remaining $N - 2$ agents.

4.1 Dependence and the distribution of fatalities

Let us consider a simple case first. Suppose that the risks between two individuals are correlated, but that these two risks are independent from the rest of the population. Then it is easy to show that more correlation across these two individuals is equivalent to a more catastrophic distribution in the N-agent world.[7] But this case is very restrictive. In the presence of N agents, the problem of comparing the distribution of fatalities is generally not obvious. The main issue is that it is not clear how to define the dependence structure between N risks. In par-

[7] To see this, let us consider a change in the correlation between agents 1 and 2, and denote $\tilde{y} := \tilde{x}_3 + ... + \tilde{x}_N$. Then we can define $\mathsf{E}[f(\tilde{d})] = \mathsf{E}[f_1(\tilde{x}_1 + \tilde{x}_2)]$, with $f_1(x) = \mathsf{E}[f(x + \tilde{y})]$. Note that the function f_1 is concave iff f is concave. As a result, the proof for the equivalence between more correlation and more catastrophic for N agents is similar as in Proposition 1 in which we simply replace f by f_1. This proof holds because of the independency between $\tilde{x}_1$ and $\tilde{x}_2$ together with $\tilde{y}$.

ticular, the dependence structure does involve more than the pairwise correlation coefficients ρ_{ij} (as in §2.2), even for Bernouilli random variables. Nevertheless, without restricting the dependence among the N-agent risks, we can obtain the following result based on two polar cases in terms of dependence.

Proposition 5. *Consider $i = 1, 2, ..., N$ agents with $p_i \in (0, 1)$. Let $\tilde{d} = \tilde{x}_1 + ... + \tilde{x}_N$ be the distribution of fatalities. Let us now introduce:*

- *the comonotonic dependence structure $\tilde{d}^c$ (also known as the maximum correlation) between the risks;*
- *and the anticomonotic dependence structure $\tilde{d}^a$ such that in all states exactly M or $M+1$ deaths occur. Formally, M is the integer number such that the expected number of fatalities is $\mu := p_1 + p_2 + ... + p_N \in [M, M+1[$. The distribution of fatalities $\tilde{d}^a$ can thus take either value: M with probability $p_M = M + 1 - \mu$, and $M+1$ with probability $1 - p_M = \mu - M$.*

Then, we know that $\tilde{d}$ is always less catastrophic than $\tilde{d}^c$, and that $\tilde{d}$ is always more catastrophic than $\tilde{d}^a$. Namely, for all f concave and all possible distributions of fatalities $\tilde{d}$:

$$\mathsf{E}[f(\tilde{d}^c)] \leqslant \mathsf{E}[f(\tilde{d})] \leqslant \mathsf{E}[f(\tilde{d}^a)] = f(M)p_M + f(M+1)p_{M+1}.$$

Proposition 5 is a generalization of Lemma 1 for $N > 2$ agents (a detailed proof is given in Appendix A.3). The most and least catastrophic distributions of fatalities, that is $\tilde{d}^c$ and $\tilde{d}^a$ respectively, are obtained when "correlation" is maximized and minimized, respectively. Meilijson and Nadas (1979) proved that maximum correlation is obtained whenever risks are comonotonic or, in non-technical terms, concentrated to specific states of the world. The least catastrophic distribution of fatalities is obtained by a generalization of the negative dependence structure in N dimensions. Yet this generalization is far from trivial. Take the example of three risks—$\tilde{x}$, $\tilde{y}$, and $\tilde{z}$—and assume that $\tilde{x}$ is negatively correlated with $\tilde{y}$ and also negatively correlated with $\tilde{z}$; then, by definition, $\tilde{y}$ and $\tilde{z}$ must be positively correlated. This simple example highlights that it is not straightforward to define what it means to say three variables are negatively correlated. Puccetti and Rüschendorf (2012) recently proposed a rearrangement algorithm, which yields the countermonotonic dependence structure of high-dimensional problems.[8]

[8] The rearrangement algorithm is based on the following idea. Denote the distribution of fatalities aggregated over all but one agent, say agent i, by $\tilde{d}_{-i} := \sum_{j \neq i} \tilde{x}_j$. The distribution of fatalities becomes more variable iff the correlation ρ_i between $\tilde{d}_{-i}$ and $\tilde{x}_i$ increases (for any agent $i = 1, 2, ..., N$). This result follows from the fact that the variability of the distribution of fatalities is related to the variance of the individual risks: $\mathsf{var}(\tilde{d}) = \mathsf{var}(\tilde{x}_i + \tilde{d}_{-i})= \mathsf{var}(\tilde{x}_i) + \mathsf{var}(\tilde{d}_{-i}) + 2\rho_i \sqrt{\mathsf{var}(\tilde{d}_{-i})}\sqrt{\mathsf{var}(\tilde{x}_i)}$.

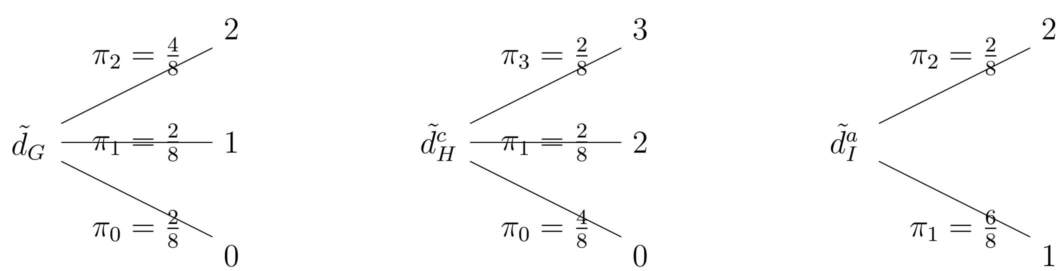
The expression of p_M is computed such that the expected number of fatalities is preserved; i.e., $p_M M + (1 - p_M)(M + 1) = \mu$. In the particular case of Proposition 5, all risks are Bernoulli distributed and the least catastrophic distribution of fatalities is therefore explicitly known (see Bernard et al. (2014) and the proof in Appendix A.3).

Let us further illustrate Proposition 5. Situations $(G) - (I)$ have the same marginal distributions for $\tilde{x}_1$, $\tilde{x}_2$, and $\tilde{x}_3$ ($p_1 = 1/2$, $p_2 = 1/4$, and $p_3 = 1/2$), but they differ with regard to their dependence structure.

$$\begin{array}{ccc} \tilde{x}_1 & \tilde{x}_2 & \tilde{x}_3 \end{array} \qquad \begin{array}{ccc} \tilde{x}_1 & \tilde{x}_2 & \tilde{x}_3 \end{array} \qquad \begin{array}{ccc} \tilde{x}_1 & \tilde{x}_2 & \tilde{x}_3 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(G) \qquad\qquad (H) \qquad\qquad (I)$$



Situation (G) is a situation with uncorrelated risks. Specifically, all pairs $\{\tilde{x}_1, \tilde{x}_2\}$, $\{\tilde{x}_1, \tilde{x}_3\}$, and $\{\tilde{x}_2, \tilde{x}_3\}$ have pairwise zero correlation ($\rho_{12} = \rho_{13} = \rho_{23} = 0$). Situation (H) gives rise to the most catastrophic distribution of fatalities $\tilde{d}^c_H$, and situation (I) to the least catastrophic distribution of fatalities $\tilde{d}^a_I$. The expected number of fatalities is in all three situations $\mu = \mathsf{E}[\tilde{\delta}] = 1.25$, but the range of possible outcomes differs across the situations. In particular, $\tilde{d}^a_I$ takes on only two values: one death with probability $p_M = 1 + 1 - 1.25 = 0.75$ and two deaths with probability $1 - p_M = 0.25$. Also observe that the least catastrophic distribution of fatalities is such that each individual risk $\tilde{x}_i$ is in reverse order (countermonotonic) with $\tilde{d}_{-i}$. In other words, the states where $\tilde{x}_i = 1$ correspond to the states of the smallest value of $\tilde{d}_{-i}$ (as outlined in footnote 8, this ensures minimum correlation). Lastly, note that the variability of the distributions of fatalities largely differs across the situations: $\mathsf{var}(\tilde{d}_G) = 11/16$ in situation (G), $\mathsf{var}(\tilde{d}^c_H) = 27/16$ in situation (H), and $\mathsf{var}(\tilde{d}_I^a) = 3/16$ in situation (I). By definition, the latter two variance terms are the maximum and minimum variance, respectively.

The result above is a mild generalization of Proposition 1. It indicates that the equivalence result between an increase in correlation and a more catastrophic distribution holds under $N > 2$ for the most extreme distributions of fatalities. However, this result is not general because it does not characterize the effect of "more dependence" in a general fashion. Moreover, we can show that another equivalence result of Proposition 1 fails. It concerns the equivalence between an increasing probability of simultaneous fatalities and a more catastrophic distribution. To show this, we simply present a counterexample with $N = 3$. Take $\tilde{d}$ defined by $(\pi_0, \pi_1, \pi_2, \pi_3) = (0, 3/5, 0, 2/5)$ and $\tilde{d}'$ by $(\pi_0, \pi_1, \pi_2, \pi_3) = (1/5, 0, 3/5, 1/5)$. Note that both distributions have the same mean (i.e., 9/5) but that $\tilde{d}$ has a higher probability of simulatenous fatalities (i.e., $2/5 > 1/5$). Nevetheless, $\tilde{d}$ has a lower probability of that at least two fatalities occur (i.e., $2/5 < 3/5$), and thus is not more catastrophic than $\tilde{d}'$.

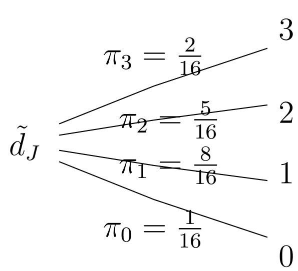
4.2 Pigou-Dalton transfers with uncorrelated risks

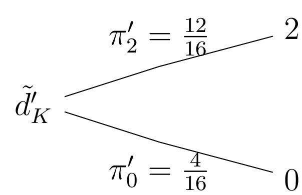
In this section, we first illustrate that pairwise correlation coefficients are insufficient to derive a necessary and sufficient condition under which a Pigou-Dalton transfer in risk would lead to a more catastrophic distribution of fatalities. We show this with the help of a simple example. Consider situations (J) and (K), below. The two situations contain $S := 16$ states of the world, in which three agents face risks ($\tilde{x}_1$, $\tilde{x}_2$, and $\tilde{x}_3$), whose pairwise correlation coefficients are equal to zero before (J) and after (K) the Pigou-Dalton transfer. One can easily verify that $corr(\tilde{x}_1, \tilde{x}_2) = corr(\tilde{x}_1, \tilde{x}_3) = corr(\tilde{x}_2, \tilde{x}_3) = 0$ and $corr(\tilde{x}'_1, \tilde{x}'_2) = corr(\tilde{x}'_1, \tilde{x}_3) = corr(\tilde{x}'_2, \tilde{x}_3) = 0$. The distributions of fatalities, $\tilde{d}_J$ and $\tilde{d}'_K$, are again depicted as trees.

$$
\begin{array}{ccc}
\tilde{x}_1 & \tilde{x}_2 & \tilde{x}_3
\end{array}
\qquad\qquad
\begin{array}{ccc}
\tilde{x}'_1 & \tilde{x}'_2 & \tilde{x}_3
\end{array}
$$

$$
\begin{bmatrix}
1 & 0 & 1 \\
1 & 1 & 1 \\
1 & 0 & 1 \\
1 & 0 & 1 \\
1 & 0 & 1 \\
1 & 1 & 1 \\
1 & 0 & 0 \\
1 & 0 & 0 \\
1 & 0 & 0 \\
1 & 1 & 0 \\
1 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}
\qquad\qquad
\begin{bmatrix}
0 & 1 & 1 \\
0 & 1 & 1 \\
1 & 0 & 1 \\
1 & 0 & 1 \\
1 & 0 & 1 \\
0 & 1 & 1 \\
1 & 1 & 0 \\
0 & 0 & 0 \\
1 & 1 & 0 \\
0 & 0 & 0 \\
1 & 1 & 0 \\
0 & 0 & 0 \\
1 & 0 & 1 \\
0 & 0 & 0 \\
0 & 1 & 1 \\
1 & 1 & 0
\end{bmatrix}
$$

(J) $\qquad\qquad$ (K)





Observe that the variance increases from $var(\tilde{d}_J) = \frac{5}{8}$ to $var(\tilde{d}'_K) = \frac{6}{8}$ due to the Pigou-Dalton transfer between agent 1 and 2, demonstrating that the post-transfer distribution of risk is more variable. However, $\tilde{d}'_K$ is not more catastrophic. Consider the concave function $f(x) = -\max(x - \psi, 0)$. For $\psi = 2.5$, $E[f(\tilde{d}'_K)] = 0 > E[f(\tilde{d}_J)]$, so that the Pigou-Dalton transfer leads to an increase in variability but not in catastrophic risk.

Situations (J) and (K) thus indicate that pairwise zero-correlation is not sufficient to maintain Keeney's result under dependent risks. Of course it can be easily checked that pairwise zero-correlation does not imply independence. Indeed, in the above example, $\tilde{x}_1$, $\tilde{x}_2$, and $\tilde{x}_3$ are not independent:

$$\Pr(\tilde{x}_1 = 1, \tilde{x}_2 = 1, \tilde{x}_3 = 1) = \frac{2}{16} \neq p_1 p_2 p_3.$$

Neither are $\tilde{x}'_1$, $\tilde{x}'_2$, and $\tilde{x}_3$:

$$\Pr(\tilde{x}'_1 = 1, \tilde{x}'_2 = 1, \tilde{x}_3 = 1) = 0 \neq p'_1 p'_2 p_3 = (p_1 - \delta)(p_2 + \delta)p_3.$$

We nevertheless can prove the following result.

Proposition 6. *Assume there are N agents facing the risks $\tilde{x}_i, ..., \tilde{x}_N$, all of which exhibit pairwise zero correlation, i.e.* $\mathsf{corr}(\tilde{x}_i, \tilde{x}_j) = 0$ *for $i \neq j$. Moreover, assume that after a Pigou-Dalton transfer in risk between agent i and j the pairwise correlation is still equal to zero:* $\mathsf{corr}(\tilde{x}'_i, \tilde{x}'_j) = 0$ *for $i \neq j$. Then,*

(i) the distribution of fatalities after the Pigou-Dalton transfer may or may not be more catastrophic;

(ii) the distribution of fatalities after the Pigou-Dalton transfer is more variable.

Proof. The counterexample above is sufficient to prove (i). To prove (ii), we compute the variance of the distribution of fatalities before and after the Pigou-Dalton transfer when all risks have pairwise zero-correlation. Before the transfer, the variance of the distribution of fatalities is given by

$$\mathsf{var}(\tilde{d}) = \mathsf{var}(\tilde{x}_1)+\mathsf{var}(\tilde{x}_2)+\mathsf{var}(\tilde{x}_3)+2\rho\sqrt{\mathsf{var}(\tilde{x}_1)}\sqrt{\mathsf{var}(\tilde{x}_2)}+2\mathsf{cov}(\tilde{x}_1, \tilde{x}_3)+2\mathsf{cov}(\tilde{x}_2, \tilde{x}_3), \tag{5}$$

where $\mathsf{var}(\tilde{x}_i) = p_i(1 - p_i)$ for $i = \{1, 2, 3\}$. After the transfer, the variance of the distribution of fatalities becomes

$$\mathsf{var}(\tilde{d}') = \mathsf{var}(\tilde{x}'_1)+\mathsf{var}(\tilde{x}'_2)+\mathsf{var}(\tilde{x}'_3)+2\rho'\sqrt{\mathsf{var}(\tilde{x}'_1)}\sqrt{\mathsf{var}(\tilde{x}'_2)}+2\mathsf{cov}(\tilde{x}'_1, \tilde{x}'_3)+2\mathsf{cov}(\tilde{x}'_2, \tilde{x}'_3), \tag{6}$$

with $\mathsf{var}(\tilde{x}'_1) = (p_1 - \delta)(1 - p_1 + \delta)$, $\mathsf{var}(\tilde{x}'_2) = (p_2 + \delta)(1 - p_2 - \delta)$, $\mathsf{var}(\tilde{x}'_3) = p_3(1 - p_3)$.

Using the information on the pairwise zero-correlation, we further simplify expressions (5) and (6) to obtain

$$\mathsf{var}(\tilde{d}) = p_1(1 - p_1) + p_2(1 - p_2) + \mathsf{var}(\tilde{x}_3),$$

$$\mathsf{var}(\tilde{d}') = (p_1 - \delta)(1 - p_1 + \delta) + (p_2 + \delta)(1 - p_2 - \delta) + \mathsf{var}(\tilde{x}'_3).$$

The above result highlights that the Pigou-Dalton transfer only affects the variances of $\tilde{x}_1$ and $\tilde{x}_2$, while $\mathsf{var}(\tilde{x}'_3) = \mathsf{var}\tilde{x}_3)$ because the distribution and the pairwise correlations of $\tilde{x}_3$ did not change. It follows that for all $\delta \in [0, \frac{p_1 - p_2}{2}]$, $\mathsf{var}(\tilde{d}') > \mathsf{var}(\tilde{d})$. The generalization of the proof to $N > 3$ is straightforward.■

Proposition 6 thus demonstrates that, for $N > 2$, it is not possible to state generic conditions about whether the distribution becomes more or less catastrophic based on information about pairwise correlations alone. Yet, we can conclude about the variability of the distribution of fatalities. In the rest of the paper, we will focus on the variability of the distribution of fatalities.

4.3 Pigou-Dalton transfers with correlated risks

In this section, we address situations in which the risks to multiple agents are correlated. Because of §4.2's inconclusive result on the degree of catastrophic risk, we focus on the variability of the distribution of fatalities.

As before, we study a situation with $N > 2$ agents at risk. A Pigou-Dalton transfer in risk is implemented between agent 1 and 2, implying that $p_1 > p_2$. If there is dependence between risks in the N-agent world, we cannot even conclude whether or not the distribution of fatalities becomes more variable due to the Pigou-Dalton transfer. The following impossibility result underpins our claim.

Proposition 7. *Consider N correlated risks $\tilde{x}_1, ..., \tilde{x}_N$. The effect of a Pigou-Dalton risk transfer from agent 1 to agent 2 on the distribution of fatalities is ambiguous in the following sense: If the dependence between agent 3,..,N can change due to the transfer, and if N is large enough, then it is generally impossible to conclude about whether the distribution becomes more or less variable.*

Proof. We study the distribution of fatalities $\tilde{d} = \tilde{x}_1 + \tilde{x}_2 + ... + \tilde{x}_N$. For notational convenience we partition the distribution into $\tilde{d} = \tilde{x}_1 + \tilde{x}_2 + \tilde{y}$ with $\tilde{y} := \tilde{x}_3 + ... + \tilde{x}_N$. It suffices to show that it is not possible to conclude whether the distribution becomes more variable when N is large enough. To do so, we compute the variance before and after the Pigou-Dalton risk transfer. The variance of the pre-transfer distribution of fatalities, $\mathsf{var}(\tilde{d})$, and the post-transfer distribution of fatalities, $\mathsf{var}(\tilde{d}')$, follows from (5) and (6), respectively. We are interested in the change in variance:

$$\mathsf{var}(\tilde{d}') - \mathsf{var}(\tilde{d}) = \Delta_{\text{PD}} + \Delta_{\text{OA}} + 2\Delta_{\mathsf{cov}},$$

which we split into three terms.

The first term is the change in variance caused by the Pigou-Dalton transfer:

$$\Delta_{\text{PD}} := \mathsf{var}(\tilde{x_1}' + \tilde{x_2}') - \mathsf{var}(\tilde{x_1} + \tilde{x_2});$$

the second term captures the change in dependence among the other agents not involved in the Pigou-Dalton transfer:

$$\Delta_{\text{OA}} := \mathsf{var}(\tilde{y}') - \mathsf{var}(\tilde{y});$$

and the third term captures the change in dependence between the two agents involved in the Pigou-Dalton transfer and all the others:

$$\Delta_{\mathsf{cov}} := \mathsf{cov}(\tilde{x_1}' + \tilde{x_2}', \tilde{y}') - \mathsf{cov}(\tilde{x_1} + \tilde{x_2}, \tilde{y}).$$

Of course the three terms are not of the same size. If N is large, the change in variance caused by the Pigou-Dalton transfer Δ_{PD} is bounded, whereas Δ_{OA} is not. Specifically, for any $p_1 > p_2$ and any $\delta \in [0, \frac{p_1 - p_2}{2}]$,

$$1 \geqslant \Delta_{\mathrm{PD}} \geqslant -2. \tag{7}$$

The proof of (7) is straightforward and thus omitted.[9] To compute Δ_{OA}, we use the two extreme cases identified in Proposition 5, in which $\tilde{y}'$ is a sum of Bernoulli variables with the respective probabilities p_j for $j = 3, ..., N$. Let $\mu_{\mathrm{OA}} := p_3 + ... + p_N \in [Z, Z+1[$ define the expected number of fatalities among the agents not involved in the transfer (with Z being an integer). Then

$$m_y \leqslant \mathsf{var}(\tilde{y}') \leqslant M_y,$$

where the definitions of minimum variance $m_y := (1 - \mu_{\mathrm{OA}} + Z)(\mu_{\mathrm{OA}} - Z)$ and maximum variance $M_y := \left(\sqrt{p_3(1-p_3)} + ... + \sqrt{p_N(1-p_N)}\right)^2$ hold for both $\mathsf{var}(\tilde{y}')$ and $\mathsf{var}(\tilde{y})$, respectively. Both extreme situations are possible in the sense that there exists a change in the dependence structure of the risks faced by agents 3, ..., N such that $\mathsf{var}(\tilde{y})$ is either equal to the minimum variance m_y; or equal to the maximum variance M_y. When $N \to \infty$, the maximum variance M_y goes to $+\infty$ and the minimum variance satisfies $m_y \in [0, 1]$ (because $0 \leqslant \mu_{\mathrm{OA}} - Z < 1$). Thus,

$$m_y - M_y \leqslant \Delta_{\mathrm{OA}} \leqslant M_y - m_y,$$

where the lower (upper) bound is obtained when $\tilde{y}$ has maximum (minimum) variance and $\tilde{y}'$ has minimum (maximum) variance. In other words, the lower bound is equal to the maximum decrease in variance due to the change in the dependence structure between the risks $\tilde{x}_3$,...,$\tilde{x}_N$ and the upper bound is the maximum change of variance due to this change in dependence.

The term Δ_{cov} may add further variability, but what is important to notice is that the change in the dependence structure of the risks to the agents not involved in the Pigou-Dalton transfer (Δ_{OA}) potentially offsets any other change in variability (Δ_{PD} or Δ_{cov} or their sum), because it is unbounded when $N \to \infty$. ■

[9](7) follows directly from Lemma 1. There are two possible ways to compute Δ_{PD}: (i) as the difference between the minimum and maximum variances between the two initial risks $\tilde{x}_1$ and $\tilde{x}_2$; or (ii) as the difference between the minimum and maximum variances between the two risks $\tilde{x}'_1$ and $\tilde{x}'_2$ after the Pigou-Dalton transfer. When $p_1 + p_2 > 1$ then the difference between the maximum variance after the transfer and the minimum variance before the transfer is equal to $2(\delta + (1 - p_1))$, which is less than unity under the assumption on the range of δ. The difference between the maximum variance before the transfer and the minimum variance after the transfer is equalt to $-2(1-p_1)$, which is larger than -2. In the case of $p_1 + p_2 \leqslant 1$ the bounds are $2(\delta + p_2)$ and -$2p_2$ respectively and the same conclusion holds.

Proposition 7 states that it is impossible to predict how a risk transfer alters the degree of variability when this transfer can also affect the dependence structure between the agents who are not directly involved in the transfer. In the following, we further constrain the problem and assume that the dependence structure of the risks not involved in the transfer is fixed. This additional hypothesis allows us to formally describe how a Pigou-Dalton transfer between two correlated risks affects the distribution of fatalities. While the equivalence results established in Proposition 8 resemble those established in Proposition 1, remember though that the notion of "more variable" is more restrictive than that of "more catastrophic" in general.

Proposition 8. *Assume that the dependence structure between the risks to agents 3,..,N is not altered by a Pigou-Dalton risk transfer of $\delta \in \left[0, \frac{p_1-p_2}{2}\right]$ between agents 1 and 2. Then, the two following statements are equivalent:*

(i) the distribution of fatalities is more variable;

(ii) the new correlation ρ' between $\tilde{x}'_1$ and $\tilde{x}'_2$ is strictly larger than a critical level of correlation, i.e.

$$\rho' \geqslant \rho^* + \frac{\mathsf{cov}\left(\tilde{x}_1 + \tilde{x}_2, \sum_{i=3}^N \tilde{x}_i\right) - \mathsf{cov}\left(\tilde{x}'_1 + \tilde{x}'_2, \sum_{i=3}^N \tilde{x}_i\right)}{\sqrt{(p_1-\delta)(1-p_1+\delta)}\sqrt{(p_2+\delta)(1-p_2-\delta)}}, \tag{8}$$

where ρ^* is the critical level of correlation (1) *given in Proposition 3 for the two-agent world.*

Proof. By assumption $\mathsf{var}(\tilde{y}) = \mathsf{var}(\tilde{y}')$. Therefore, $\mathsf{var}(\tilde{d}') > \mathsf{var}(\tilde{d})$ iff

$$\begin{aligned}&\mathsf{var}(\tilde{x}'_1) + \mathsf{var}(\tilde{x}'_2) + 2\rho'\sqrt{\mathsf{var}(\tilde{x}'_1)}\sqrt{\mathsf{var}(\tilde{x}'_2)} + 2\mathsf{cov}(\tilde{x}'_1 + \tilde{x}'_2, \tilde{y})\\ &\qquad > \mathsf{var}(\tilde{x}_1) + \mathsf{var}(\tilde{x}_2) + 2\rho\sqrt{\mathsf{var}(\tilde{x}_1)}\sqrt{\mathsf{var}(\tilde{x}_2)} + 2\mathsf{cov}(\tilde{x}_1 + \tilde{x}_2, \tilde{y}).\end{aligned}$$

Solving for ρ' yields:

$$\rho' > \frac{\delta\ (\delta - p_1 + p_2)}{\sqrt{\mathsf{var}(\tilde{x}'_1)}\sqrt{\mathsf{var}(\tilde{x}'_2)}} + \rho\frac{\sqrt{\mathsf{var}(\tilde{x}_1)}\sqrt{\mathsf{var}(\tilde{x}_2)}}{\sqrt{\mathsf{var}(\tilde{x}'_1)}\sqrt{\mathsf{var}(\tilde{x}'_2)}} + \frac{\mathsf{cov}(\tilde{x}_1 + \tilde{x}_2 - \tilde{x}'_1 - \tilde{x}'_2, \tilde{y})}{\sqrt{\mathsf{var}(\tilde{x}'_1)}\sqrt{\mathsf{var}(\tilde{x}'_2)}},$$

where the first two terms on the LHS are equal to the critical level of correlation ρ^* found in Proposition 3 for $N = 2$. ■

Some observations on Proposition 8 are warranted. First, note that for the special case analyzed by Keeney (1980), we have $\rho = \rho' = cov(\tilde{x}_1+\tilde{x}_2-\tilde{x}'_1-\tilde{x}'_2, \tilde{y}) = 0$ so that the inequality (8) is satisfied because $\delta - p_1 + p_2 < 0$. More generally, the presence of N agents does not alter the result obtained with only two agents whenever

$$\mathsf{cov}\left(\tilde{x}_1+\tilde{x}_2,\sum_{i=3}^{N}\tilde{x}_i\right)=\mathsf{cov}\left(\tilde{x}'_1+\tilde{x}'_2,\sum_{i=3}^{N}\tilde{x}_i\right).$$

In words, Proposition 8 holds when the Pigou-Dalton risk transfer between agent 1 and 2 does not affect the correlation between the risks faced by these two agents and the risks faced by the other $j=3,...,N$ agents.

5 Conclusion

In this paper, we examine the statistical dependence of social risks. We ask: What is the relationship between more catastrophic, more correlated and more equitable risks? We define a more catastrophic social situation as a mean-preserving spread of the distribution of fatalities, and a more equitable sitation as a risk transfer that reduces in the difference in the probability of dying of two agents. We show that a higher correlation between two individual risks is always equivalent to a more catastrophic situation, as well as to a higher probability of simultaneous deaths. We also characterize a set of conditions under which risk equity induces a more catastrophic situation. In particular, these conditions hold when a change in risk equity does not decrease the correlation between the two risks, or when one individual's death or survival is certain. Our paper therefore identifies the conditions so that the ex ante / ex post conflict in risk management exhibited by Keeney (1980) generally carries over to a world with dependent risks. Note that we allow for risk dependency to vary before and after the equity-increasing transfer. We believe that this degree of generality is important because some new technologies may change both risk equity and risk dependency in the society. The extension of these results to the N-agent world is complex, in particular because (pairwise) correlation is not enough to characterize the dependence of social risks. We can however generalize some of the results in a N-agent world by assuming that the risk transfer between two agents does not affect the dependence structure among the remaining $N-2$ agents.

This paper contributes to the literature on the management of catastrophic risks. Given the increasing concerns toward global risks such as climate change or large pandemia, this topic has attracted a lot of attention recently. Although some scholars argue that we should focus more on the prevention of extreme risks (Posner 2004, Weitzman 2009), others argue that we focus too much on low probability but salient risks (Sunstein 2005). Interestingly, recent work in social choice can axiomatize either catastrophe averse or catastrophe prone preferences (Bommier and Zuber 2008, Fleurbaey 2010). Moreover, some empirical studies indicate that neither experts nor lay people are catastrophe averse (Jones-Lee and Loomes 1995, Rheinberger 2010). In practice, cost benefit analysis evaluates monetary-equivalent changes in mortality by computing the expected number of lives saved multiplied by the value of statistical life. Thus cost benefit analysis is insensitive to the catastrophic aspect of social risks. Yet, some regulatory agencies display a form of catastrophe aversion by considering quantitative metrics which increase with the size of accidents (Bedford 2013). However, these metrics are not well grounded conceptually, and it is not clear how they should be combined with alternative criteria. Obviously, there exist many criteria to be considered by risk managers and policy makers. A natural research agenda thus concerns the generalization of standard risk management and policy evaluation tools to make them consistent with a broader set of risk criteria including preferences toward risk equity and catastrophic risks.

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A Appendix

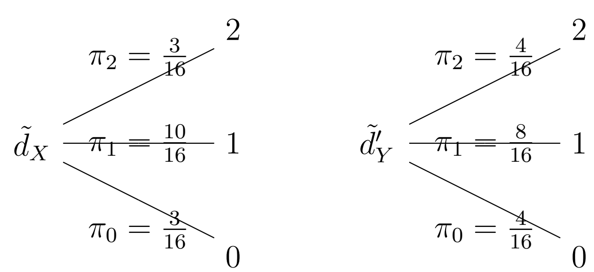
A.1 Example of ex-post independent risks

We asserted in §3.1 that it is possible to find examples in which the two risks before the Pigou-Dalton transfer, $\tilde{x}_1$ and $\tilde{x}_2$, as well as after the transfer, $\tilde{x}'_1$ and $\tilde{x}'_2$, are uncorrelated and hence independent.[10] Consider the below situations (X) and (Y) with $S := 16$ states $(\omega_1, ..., \omega_{16})$.

$$
\begin{array}{cc}
\tilde{x}_1 & \tilde{x}_2
\end{array}
\qquad\qquad
\begin{array}{cc}
\tilde{x}'_1 & \tilde{x}'_2
\end{array}
$$

$$
\begin{bmatrix}
1 & 1 \\
1 & 1 \\
1 & 1 \\
0 & 1 \\
1 & 0 \\
1 & 0 \\
1 & 0 \\
0 & 0 \\
1 & 0 \\
1 & 0 \\
1 & 0 \\
0 & 0 \\
1 & 0 \\
1 & 0 \\
1 & 0 \\
0 & 0
\end{bmatrix}
\qquad\qquad
\begin{bmatrix}
1 & 1 \\
1 & 1 \\
0 & 1 \\
0 & 1 \\
1 & 1 \\
1 & 1 \\
0 & 1 \\
0 & 1 \\
1 & 0 \\
1 & 0 \\
0 & 0 \\
0 & 0 \\
1 & 0 \\
1 & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix}
$$

$$(X) \qquad\qquad (Y)$$



The individual risks $\tilde{x}_1$ and $\tilde{x}_2$ in the initial situation (X) are independent, and so are the individual risks $\tilde{x}'_1$ and $\tilde{x}'_2$ after a Pigou-Dalton transfer of $\delta = \frac{1}{4}$.

[10] In settings with more than two risks, zero correlation does not necessarily imply independence (see situations (J) and (K) in §4.2). Yet for two Bernoulli random variables zero correlation always implies independence.

When the risks are independent before and after the Pigou-Dalton transfer, as in situations (X) and (Y), the distribution of fatalities $\tilde{d}' = \tilde{x}'_1 + \tilde{x}'_2$ is more catastrophic than $\tilde{d} = \tilde{x}_1 + \tilde{x}_2$ as predicted by Keeney (Proposition 2).

A.2 Proof of statement (i) in Proposition 4 when $\rho < 0$

From (1) we have that

$$\rho^*(\delta) := \frac{\delta(p_2 - p_1 + \delta) + \rho\sqrt{p_1}\sqrt{1-p_1}\sqrt{p_2}\sqrt{1-p_2}}{\sqrt{1-p_1+\delta}\sqrt{p_1-\delta}\sqrt{p_2+\delta}\sqrt{1-p_2-\delta}}.$$

We compute the derivative of this value with respect to δ:

$$\frac{\partial\rho^*(\delta)}{\partial\delta} = \frac{-1}{2}\frac{(p_2 - p_1 + 2\delta)(A(\delta)\rho + B(\delta))}{[(p_1-\delta)(1-p_1+\delta)(1-p_2-\delta)(p_2+\delta)]^{\frac{3}{2}}} \tag{9}$$

where

$$A(\delta) := \sqrt{p_1(1-p_1)p_2(1-p_2)}(2\delta^2 + 2\delta(p_2 - p_1) - (1 + 2p_1p_2 - p_2 - p_1))$$

and

$$B(\delta) := (2p_1p_2 + 1 - p_1 - p_2)(\delta^2 + \delta(p_2 - p_1) - 2p_1p_2) + 2p_1^2p_2^2.$$

Lemma 2. *The function $\delta \mapsto A(\delta)\rho + B(\delta)$ satisfies the following property*

$$\forall\delta \geqslant 0, \quad A(\delta)\rho + B(\delta) < 0. \tag{10}$$

Proof of Lemma 2. Observe that from the expressions of $A(\delta)$ and $B(\delta)$, we know that the critical value ρ^* is a second-degree polynomial of δ whose limit goes to $-\infty$ if $\delta \to \infty$ (because $\rho < 0$). In addition, we now prove that

$$A(0)\rho + B(0) < 0 \quad A'(0)\rho + B'(0) < 0. \tag{11}$$

Thus (10) follows because of the properties of a polynomial of the second degree in δ. To prove (11), we need the expressions of $A(0)$, $B(0)$, $A'(0)$ and $B'(0)$:

$$\begin{aligned}
A(0) &= -\sqrt{p_1(1-p_1)p_2(1-p_2)}(p_1p_2 + (1-p_1)(1-p_2)) \\
B(0) &= -2p_1p_2(1-p_1)(1-p_2) \\
A'(0) &= 2(p_2-p_1)\sqrt{p_1(1-p_1)p_2(1-p_2)} \\
B'(0) &= (p_2-p_1)(p_1p_2 + (1-p_1)(1-p_2)).
\end{aligned}$$

We distinguish two cases:
Case 1: when $p_1 + p_2 \leqslant 1$. Then from Lemma 1, $\rho \geqslant \frac{-\sqrt{p_1p_2}}{\sqrt{1-p_1}\sqrt{1-p_2}}$ then

$$A(0) + \rho B(0) \leqslant p_1p_2(p_1 + p_2 - 1) \leqslant 0$$

$$A'(0) + \rho B'(0) \leqslant (p_2 - p_1)(1 - p_1 - p_2) < 0$$

because $p_2 < p_1$.
Case 2: when $p_1 + p_2 > 1$. Then from Lemma 1, $\rho \geqslant \frac{-\sqrt{1-p_1}\sqrt{1-p_2}}{\sqrt{p_1 p_2}}$ then

$$A(0) + \rho B(0) \leqslant (1 - p_1)(1 - p_2)(1 - p_1 + p_2) < 0$$

$$A'(0) + \rho B'(0) \leqslant (p_2 - p_1)(p_1 + p_2 - 1) < 0$$

because $p_2 < p_1$. ■

Proof of statement (i) in Proposition 4 when $\rho < 0$. Since $\delta \leqslant \frac{p_1 - p_2}{2}$, then from the expression (9) and Lemma 2, it is clear that

$$\forall \delta \in \left[0, \frac{p_2 - p_1}{2}\right] \qquad \frac{\partial \rho^*(\delta)}{\partial \delta} \leqslant 0. \tag{12}$$

When $\delta = 0$, then $\rho^* = \rho$. Using the fact that ρ^* is decreasing in δ (i.e. (12)), then for all $\delta \in \left[0, \frac{p_2 - p_1}{2}\right]$ $\rho^*(\delta) \leqslant \rho^*(0) = \rho \leqslant \rho'$. Since ρ' satisfies (1), then the distribution of fatalities is more catastrophic and (i) is proved. ■

A.3 Proof of statement (ii) in Proposition 5

The proof of (ii) in Proposition 5 is inspired from Lemma 3.1 in Bernard et al. (2014b). We prove here that the resulting distribution is more catastrophic and not only more variable as it is the case in their paper.

Lemma 3 (Least catastrophic distribution of fatalities). *Define for $j = 1, \ldots, N$,*

$$a_j = \left(\sum_{i=1}^{j} p_i\right) \bmod 1,$$

and the sets

$$I_j = \begin{cases} [a_{j-1}, a_j] & \text{if } \ a_j > a_{j-1} \\ [0, a_j] \cup [a_{j-1}, 1] & \text{if } \ a_j < a_{j-1} \end{cases},$$

where we define $a_0 = 0$. Then, the least catastrophic distribution is $\tilde{d}^a := \sum_{j=1}^{N} \tilde{y}_j$ where $\tilde{y}_j$ are defined by

$$\tilde{y}_j = \mathbb{1}_{\tilde{u} \in I_j}, \tag{13}$$

where $\tilde{u}$ is a standard uniformly distributed random variable over (0,1). Furthermore, $\tilde{d}^a$ takes only two values M with probability $p_M = M + 1 - \mu$ and $M + 1$ with probability $1 - p_M$ where $M = \lfloor \mu \rfloor$ (largest integer inferior or equal to μ).

Proof. Let us first observe that $\tilde{y}_j$ defined by (13) are Bernoulli with parameter p_j.

Furthermore, $\tilde{d}^a = \tilde{y}_1 + \tilde{y}_2 + \cdots + \tilde{y}_N$ only takes values M with probability p_M or $(M+1)$ with probability $1-p_M$ (where $p_M = 1$ may hold if it is constant). Consider any other distribution of fatalities $\tilde{d} = \tilde{x}_1 + \tilde{x}_2 + \cdots + \tilde{x}_N$ with $\tilde{x}_j$ being a Bernoulli distribution with parameter p_j and let us show that $\tilde{d}$ is more catastrophic than $\tilde{d}^a$. Observe that any such distribution of fatalities $\tilde{d}$ takes values in $\{0, 1, 2, \ldots, N\}$.

It is clear that $\forall x \in]0, M[$, $F_{\tilde{d}}(x) \geqslant F_{\tilde{d}^a}(x) = 0$ and $\forall x \in [(M+1), +\infty[$, $F_{\tilde{d}}(x) \leqslant F_{\tilde{d}^a}(x) = 1$. Since $F_{\tilde{d}}(x)$ and $F_{\tilde{d}^a}(x)$ are constant on the interval $[M, M+1[$ one has

$$\exists c \geqslant 0, \quad \begin{cases} \forall x \in (0, c), & F_{\tilde{d}}(x) \geqslant F_{\tilde{d}^a}(x) \\ \forall x \in (c, +\infty), & F_{\tilde{d}}(x) \leqslant F_{\tilde{d}^a}(x) \end{cases} \tag{14}$$

namely, $c = M + 1$ if $F_{\tilde{d}}(M) > F_{\tilde{d}^a}(x)$ and $c = M$ if $F_{\tilde{d}}(M) \leqslant F_{\tilde{d}^a}(x)$. In other words, the distribution function $F_{\tilde{d}}$ crosses $F_{\tilde{d}^a}$ exactly once from above. Since $E(\tilde{d}) = E(\tilde{d}^a)$ this implies the well-known one-crossing property that characterizes second-order stochastic dominance. ■