An analysis of insurance in the newsboy problem

Richard Watt and Francisco J. Vazquez

July 2, 2015

Abstract

In this paper we study the standard newsboy problem, but under two new assumptions when compared to the existing literature. First, we assume that the wholesaler is an expected profit maximiser who sets the wholesale price optimally, and in doing so, takes into account the salvage value at which the newsboy can return unsold items to the wholesaler. Second, we assume that the salvage value is a choice variable of the newsboy, and in that way, it acts as a standard insurance device. The newsboy's optimal salvage value then represents an optimal demand for insurance. We study in particular the optimal pricing problem of the supplier, and show that it can be expressed as a mark-up equation. We also show that insurance is provided at an actuarially unfair price. As regards the optimal demand for insurance by the newsboy, the problem is too complex for a closed form solution to be possible, so we resort to a simulation which returns the results that a strictly positive level of strictly partial insurance is demanded when the newsboy is strictly risk averse, and the optimal level of insurance coverage increases with risk aversion.

1 Introduction

In this paper we consider the classic newsboy inventory problem, with a weakly risk averse newsboy. The introduction of risk aversion to the problem is, of course, not new, with the seminal paper having been published more than 40 years ago (Baron, 1973), and the most complete analysis having been published 20 years ago (Eeckhoudt et al., 1995).[1] There is, however, one aspect of the risk averse newsboy problem that does not (to the best of our knowledge) appear to have been studied, and that is the question of insurance.[2] Not withstanding the lack of attention to insurance, a clear insurance mechanism has been a standard inclusion in many papers, namely a salvage value for unsold units. In the present paper we focus on using the salvage value as an insurance device, and we look at the optimal insurance contracts using that device.

As an integral part of the paper, we also allow not only the newsboy to make optimal choices, but also the supplier. Specifically, we assume that the supplier sets the wholesale price of units sold to the newsboy optimally, with the objective of maximizing expected profit given the newsboy's choices. This is also an aspect of the newsboy problem that has not been included in many previous papers, where the supplier exists only as an exogenous parameter set, normally only the wholesale price that the newsboy

[1] Several other papers that include a risk averse newsboy have appeared since then (e.g. Agrawal and Seshadri 2000, and Wang et al. 2009). A detailed survey can be found in Wang et al. (2012).

[2] Of course, there are plenty of papers that have looked at risk-sharing between the newsboy and the supplier (e.g.Tsay 2002, Gan et al. 2004, and Cachon and Lariviere 2005). But no paper thus far explicitly considers the relationship between the newsboy and the supplier in terms of insurance.

purchases at.[3] It is, however, an integral part of the present paper, as it allows us to study a proper market environment in which insurance, as represented by the salvage value for unsold items, becomes a choice variable for the newsboy. Other papers have made the salvage mechanism a choice variable for the supplier; see, for example, Pasternack (1985), and Lau and Lau (1999). We discuss this option below, but we note that since our objective is to consider insurance, we are more interested in allowing the insurance consumer (the newsboy) choose his optimal insurance demand, given the conditions set by the insurance provider (here, the supplier).[4] We do not allow the supplier to charge separately for the insurance mechanism that is represented by the salvage value re-purchase, although we are very interested in how the insurance mechanism affects the optimal supplier price for units of product sold to the newsboy.

In the paper we consider the comparative statics of the optimal level of insurance, r^*, as chosen by the newsboy. In particular, we are interested in how insurance demand changes with the newsboy's risk aversion. Notice that, in principle, the effect is ambiguous, since not only does risk aversion affect the optimal desire for insurance, it will also affect the optimal per-unit price that the supplier charges.

While we use the newsboy problem to study both optimal supplier choices along with optimal demander (in our case, the newsboy) choices when insurance is an integral aspect, there are other senarios that also adhere to the same general scheme. Another clear example of the same is when a retailer sells a good along with a guarantee of some sort, but where the guarantee is not charged for separately from the actual good.

2 Assumptions

The supplier charges a wholesale price of c for each unit sold to the newsboy, and offers to buy back unsold units at a salvage value of r dollars per unit. The newsboy sells units to the market at an exogenously determined retail price p. We restrict $0 \leq r \leq c \leq p$. For reasons set out below, we also assume $r < p$. We assume that the supplier's cost function is restricted to a constant marginal cost of q per unit supplied to the newsboy, with no fixed cost.

Retail sales are given by x, which is distributed according to pdf $f(x)$ and distribution $F(x)$. This density and distribution would be changed by a change in the retail price p, but we will hold p constant in all that we do, and so we avoid making arbitrary assumptions about f as a function of p. Essentially, we simply take the retail price as given (which of course is not a bad assumption for the particular case of newspapers, which normally are sold at a given price each day), and take the probability density function to be that corresponding to the given retail price. We normalise our units so that sales are defined on the unit interval, i.e. $0 \leq x \leq 1$. Of course, the newsboy must choose his order quantity, y, before the realization of demand.

We interpret $r = 0$ as no insurance, $r = c$ as full insurance, and $0 < r < c$ as partial insurance. Of course other types of insurance could easily be modelled. For example, the newsboy might be given a choice of how many units of the order are able to be re-sold back to the supplier at a set price of r. But in the present paper we concentrate only upon the simpler case of all unsold units being refunded. We firstly study a base-line model in which r is an exogenous parameter, in order to see how r affects the optimal choices of c (by the supplier) and of y (by the newsboy), and then we tackle the problem of allowing the newsboy to

[3] Notable exceptions are Pasternack (1985) for the case of a risk neutral newsboy, and with risk aversion, Lau and Lau (1999) and Lariviere and Porteus (2001). However, these papers are mainly concerned with supply chain coordination rather than supplier profit maximization. In unpublished work, Tang and Rudi (2010) have a very interesting paper that does look at optimal supplier pricing under the objective of maximization of expected profit.

[4] Cachon and Kök (2007) provide a model in which the newsboy does implicitly choose the insurance conditions, but instead of being insured by the supplier, the insurance happens directly in the market for the good, by allowing items unsold in the first instance to be offered at a discount later on.

set both y and r.

The model works as follows. The supplier sets the wholesale price, c, under full information on the newsboy's utility and the other parameters in the model. The newsboy then takes c and makes an optimal choice of either just the order quantity, y (in the baseline model), or both the order quantity and the salvage value price, r (in the insurance demand model). Given the assumption of full information, of course the supplier backward inducts (i.e. acts as a Stackelberg leader), and uses the newsboy's optimal decisions when making her own choice of the optimal wholesale price.

Since we are interested in insurance, we assume that the newsboy is not risk loving - he is either risk averse or risk neutral, or more sucinctly, the newsboy is weakly risk averse. The newsboy's utility function for wealth w is $u(w)$, where we assume $u'(w) > 0$ and $u''(w) \leq 0$. The newsboy is assumed to maximize expected utility. On the other hand, we assume that the supplier is risk neutral, and maximizes her expected monetary profit.

3 The newsboy's problem

While now well-known in the literature, we firstly review the expected utility maximization problem of the newsboy under the assumption that r is not a choice variable. Given c and r, the newsboy's expected utility of an order of size y is

$$\begin{aligned} Eu &= \int_0^y u\left(x\left(p-c\right)+\left(y-x\right)\left(r-c\right)\right)f(x)dx + \int_y^1 u\left(y\left(p-c\right)\right)f(x)dx \\ &= \int_0^y u\left(x\left(p-c\right)+\left(y-x\right)\left(r-c\right)\right)f(x)dx + u\left(y\left(p-c\right)\right)\left[1-F(y)\right] \end{aligned}$$

Define $w_1 = x\left(p-c\right)+\left(y-x\right)\left(r-c\right)$ and $w_2 = y\left(p-c\right)$, where w_1 is defined on $0 \leq x \leq y$ and w_2 on $y \leq x \leq 1$. Notice that w_2 is equal to w_1 when $x = y$.

The first-order condition for an optimal order (given r and c) is

$$\frac{\partial Eu}{\partial y} = \int_0^{y^*} u'(w_1^*)\left(r-c\right)f(x)dx + u'(w_2^*)\left(p-c\right)\left[1-F(y^*)\right] = 0 \tag{1}$$

The second derivative of expected utility is

$$\begin{aligned} \frac{\partial^2 Eu}{\partial y^2} &= u'(w_2^*)(r-c)f(y^*) + \int_0^{y^*} u''(w_1^*)\left(r-c\right)^2 f(x)dx \\ &\quad + u''(w_2^*)\left(p-c\right)^2\left[1-F(y^*)\right] - u'(w_2^*)\left(p-c\right)f(y^*) \\ &= u'(w_2^*)(r-p)f(y^*) + \int_0^{y^*} u''(w_1^*)\left(r-c\right)^2 f(x)dx \\ &\quad + u''(w_2^*)\left(p-c\right)^2\left[1-F(y^*)\right] \end{aligned}$$

The second-order condition (that expected utility is concave in y) is satisfied by non-convexity of the utility function. In passing, note that even if utility is linear, then if $r - p < 0$, expected utility is still concave in y.

Clearly y^* is a function of both c and r, so we can write $y^*(c, r)$. We note the following:

$$\begin{aligned}
r &< c = p \quad \Rightarrow \quad y^* = 0 \\
r &= c < p \quad \Rightarrow \quad y^* = 1 \\
r &= c = p \quad \Rightarrow \quad y^* \text{ is indeterminate}
\end{aligned}$$

The indeterminacy with $r = c = p$ is important to the model, and it happens because the newsboy would earn 0 profit from any choice of order. If it happens that the supplier could just as easily as the newsboy gain access to the market, then the option of setting $r = c = p$ essentially recognises that the newsboy does not contribute anything to the value of the business, and by using $r = c = p$ the supplier captures all of the market surplus as profit. It is as if the supplier himself becomes the newsboy, and is able to capture the entire maximum amount of profit from the market.[5] So if we allow the supplier to set both c and r, this will be the outcome that we get. The newsboy will be fully insured, but will be charged a unit price equal to the market price and will therefore earn zero expected profit. The supplier will recieve the entire profit from the business.[6] In order to remove this indeteminacy as a possibility, we henceforth restrict the parameters such that $r < p$.

It is instructive to consider the comparative statics of the optimal order as r and c change.[7] As noted, this is only interesting if $0 \leq r < p$. Let's begin with an increase in r. From the implicit function theorem, since the second-order condition is satisfied, we know that

$$\text{sign}\frac{\partial y^*}{\partial r} = \text{sign}\frac{\partial^2 Eu}{\partial y \partial r}$$

Deriving (1) with respect to r, we find that

$$\frac{\partial^2 Eu}{\partial y \partial r} = \int_0^{y^*} \left(u'(w_1^*) + u''(w_1^*)\,(y^* - x)\,(r - c)\right) f(x)dx > 0$$

where the sign is due to the fact that both $u''(w_1^*)$ and $r - c$ are negative. So we know that an increase in the level of insurance through the refund scheme leads to a greater optimal order, for a given wholesale price c.

Second, from the implicit function theorem we know that

$$\text{sign}\frac{\partial y^*}{\partial c} = \text{sign}\frac{\partial^2 Eu}{\partial y \partial c}$$

[5] On the other hand, if the newsboy is an essential part of the supply chain, in that without the newsboy the supplier cannot access the market, then the newsboy does hold some bargaining power, and would not be restricted to expected profit of 0.

[6] Of course, this is also a Pareto efficient risk sharing arrangement, although not the only one, since the risk neutral participant retains all of the risk.

[7] Both of the comparative statics changes that we consider here were also studied, along with a host of others, by Eeckhoudt et al. (1995).

Deriving (1) with respect to c, we find that

$$\begin{aligned}\frac{\partial^2 Eu}{\partial y \partial c} &= \int_0^{y^*} \left(u''(w_1^*)\left(-x-y^*+x\right)(r-c)-u'(w_1^*)\right) f(x)dx \\ &\quad + \left(u''(w_2^*)\left(-y^*\right)(p-c)-u'(w_2^*)\right)\left[1-F(y^*)\right] \\ &= \int_0^{y^*} \left(u''(w_1^*)\left(-y^*\right)(r-c)-u'(w_1^*)\right) f(x)dx \\ &\quad + \left(u''(w_2^*)\left(-y^*\right)(p-c)-u'(w_2^*)\right)\left[1-F(y^*)\right]\end{aligned}$$

The first summand is non positive since both $u''(w_1^*)\,(-y)\,(r-c)$ and $-u'(w_1^*)$ are non-positive.[8] However, the second summand has ambiguous sign, since $u''(w_2^*)\,(-y)\,(p-c) \geq 0$ and $-u'(w_2^*) < 0$. Hence, without further assumptions we cannot say for sure exactly how an increase in the wholesale price c affects the optimal order for a given level of r. This indeterminacy is also noted in Eeckhoudt et al. (1995), who assert (without formal proof, but with a convincing intuitive argument) that a sufficient condition for the newsboy's demand to be a decreasing function of the wholesale price is that his utility function exhibits DARA. Another possibility is suggested directly from the above equation. If $u''(w_2^*)\,(-y)\,(p-c) - u'(w_2^*) = -u''(w_2^*)w_2^* - u'(w_2^*) \leq 0$, then an increase in c would have a non-positive effect upon y^*. This condition rearranges directly to the requirement that relative risk aversion at w_2^*, $\frac{-u''(w_2^*)w_2^*}{u'(w_2^*)}$, be less than or equal to 1. While a reasonably common restriction, it is unfortunately not particularly realistic, with most estimates of relative risk aversion being greater than 1. Of course, if the newsboy is risk neutral, with constant marginal utility equal to u', then it is straightforward to note that $\frac{\partial^2 Eu}{\partial y \partial c} = -u' < 0$. Thus risk-neutrality also delivers a strictly decreasing demand curve. However, interestingly, it cannot be generally stated that the newsboy's demand curve for units from the supplier is everywhere downward sloping.

That said, we already noted above that at the two extreme values of c, we get extreme solutions for y^*. Specifically, $r < c = p \Rightarrow y^* = 0$, and $r = c < p \Rightarrow y^* = 1$. A quick sketch then reveals that if demand is upward sloping somewhere, it must happen for an intermediate set of values of c, which would require a rather strange utility function to generate such a demand profile. Concretely, if $r = c < p$, so that $y^* = 1$ we get

$$\left.\frac{\partial^2 Eu}{\partial y \partial c}\right|_{r=c<p} = -\int_0^1 u'(w_1^*)f(x)dx < 0$$

Likewize, if $r < c = p$ so that $y^* = 0$, we get

$$\left.\frac{\partial^2 Eu}{\partial y \partial c}\right|_{r<c=p} = -u'(0) < 0$$

So at each extreme, the demand curve is unambiguously downward sloping.

Recall that the newsboy's optimal order, $y^*(c, r)$, satisfies

$$(r-c)\int_0^{y^*} u'(w_1^*)f(x)dx + u'(w_2^*)\,(p-c)\left[1-F(y^*)\right] = 0$$

[8] Recall that, even though we are here assuming that r is strictly less than c, we still want to allow utility to be possibly linear.

Integrating the first term by parts, we have

$$\begin{aligned}
(r-c)\left(u'(w_2^*)F(y^*) - \int_0^{y^*} u''(w_1^*)F(x)dx\right) + u'(w_2^*)(p-c)\left[1-F(y^*)\right] &= 0 \\
u'(w_2^*)F(y^*)(r-c-p+c) - (r-c)\int_0^{y^*} u''(w_1^*)F(x)dx + u'(w_2^*)(p-c) &= 0 \\
(c-r)\int_0^{y^*} u''(w_1^*)F(x)dx - u'(w_2^*)F(y^*)(p-r) + u'(w_2^*)(p-c) &= 0 \\
(c-r)\int_0^{y^*} u''(w_1^*)F(x)dx - u'(w_2^*)\left(F(y^*)(p-r)-(p-c)\right) &= 0
\end{aligned}$$

So in the end, the first-order condition can be written as

$$(c-r)\int_0^{y^*} u''(w_1^*)F(x)dx = u'(w_2^*)\left(F(y^*)(p-r)-(p-c)\right) \tag{2}$$

Clearly, so long as $c > r$ (as we are assuming), and if the newsboy is strictly risk averse, then the left-hand-side is strictly negative. Therefore, the optimal order must satisfy

$$F(y^*)(p-r)-(p-c) < 0$$

or

$$F(y^*) < \frac{p-c}{p-r}$$

It is well known that if the newsboy were risk neutral, then the optimal order, say y^{*n}, satisfies $F(y^{*n}) = \frac{p-c}{p-r}$, and so the optimal risk averse order is such that $F(y^n) > F(y^*)$, or since F is strictly increasing, $y^n > y^*$. So risk aversion reduces the optimal order as compared to risk neutrality. It is also relatively simple to show that increased risk aversion leads to a decrease in the optimal order (see, for example, Eeckhoudt et a. 1995).

3.0.1 The shape of indirect expected utility

There are (so far in our modelling), three parameters in the newsboy's decision problem, namely c, r and p. Here we only investigate the shape of indirect expected utility in c and in r, as throughout we assume p to be constant. We also always assume that the utility function is such that $\frac{\partial y^*}{\partial r} > 0 > \frac{\partial y^*}{\partial c}$ (e.g. utility is DARA would suffice).

It seems logical that, for given values of c and p, the newsboy would prefer that r be set as high as possible. However, since y^* increases with r, it is not possible to prove this assertion (at least without further assumptions on utility). That is, it may not be true that the newsboy prefers the highest possible value of r. We can see this by considering the shape of expected utility in r, i.e. the shape of indirect expected utility.

$$\begin{aligned}
Eu &= \int_0^y u\left(x(p-c)+(y-x)(r-c)\right)f(x)dx + \int_y^1 u\left(y(p-c)\right)f(x)dx \\
&= \int_0^y u\left(x(p-c)+(y-x)(r-c)\right)f(x)dx + u\left(y(p-c)\right)\left[1-F(y)\right]
\end{aligned}$$

At the optimal order we get indirect expected utility of;

$$\begin{aligned} Eu &= \int_0^{y^*} u\left(x\left(p-c\right)+\left(y^*-x\right)\left(r-c\right)\right)f(x)dx+u\left(y^*\left(p-c\right)\right)\left[1-F(y^*)\right] \\ &= \int_0^{y^*} u\left(w_1^*\right)f(x)dx+u\left(w_2^*\right)\left[1-F(y^*)\right] \end{aligned}$$

The derivative with respect to r is

$$\begin{aligned} \frac{\partial Eu}{\partial r} &= u\left(w_2^*\right)f(y^*)\frac{\partial y^*}{\partial r}+\int_0^{y^*} u'\left(w_1^*\right)\left(\left(r-c\right)\frac{\partial y^*}{\partial r}+y^*-x\right)f(x)dx \\ &\quad +u'\left(w_2^*\right)\left(p-c\right)\frac{\partial y^*}{\partial r}\left[1-F(y^*)\right]-u\left(w_2^*\right)f(y^*)\frac{\partial y^*}{\partial r} \\ &= \int_0^{y^*} u'\left(w_1^*\right)\left(\left(r-c\right)\frac{\partial y^*}{\partial r}+y^*-x\right)f(x)dx \\ &\quad +u'\left(w_2^*\right)\left(p-c\right)\frac{\partial y^*}{\partial r}\left[1-F(y^*)\right] \\ &= \left(\left(r-c\right)\frac{\partial y^*}{\partial r}+y^*\right)\int_0^{y^*} u'\left(w_1^*\right)f(x)dx-\int_0^{y^*} u'\left(w_1^*\right)xf(x)dx \\ &\quad +u'\left(w_2^*\right)\left(p-c\right)\frac{\partial y^*}{\partial r}\left[1-F(y^*)\right] \end{aligned}$$

which is of ambiguous sign (the first term may be positive or negative since $\left(r-c\right)\frac{\partial y^*}{\partial r} < 0 < y^*$, the second term is negative, and the third is positive). One thing we can do is to evaluate the derivative at the right-extreme value of r, namely $r=c$, in which case we know that we get $y^*=1$, and so

$$\begin{aligned} \left.\frac{\partial Eu}{\partial r}\right|_{r=c} &= \int_0^1 u'\left(w_1^*\right)f(x)dx-\int_0^1 u'\left(w_1^*\right)xf(x)dx \\ &= \int_0^1 u'\left(w_1^*\right)(1-x)f(x)dx>0 \end{aligned}$$

So at least when r is high enough, the newsboy's expected utility is increasing in r.

We may also look at the shape of indirect expected utility in c:

$$\begin{aligned} Eu &= \int_0^{y^*} u\left(x\left(p-c\right)+\left(y^*-x\right)\left(r-c\right)\right)f(x)dx+u\left(y^*\left(p-c\right)\right)\left[1-F(y^*)\right] \\ &= \int_0^{y^*} u\left(w_1^*\right)f(x)dx+u\left(w_2^*\right)\left[1-F(y^*)\right] \end{aligned}$$

The derivative is

$$\begin{aligned} \frac{\partial Eu}{\partial c} &= u\left(w_2^*\right)f(y^*)\frac{\partial y^*}{\partial c}+\int_0^{y^*} u'\left(w_1^*\right)\left(-x-y^*+x+(r-c)\frac{\partial y^*}{\partial c}\right)f(x)dx \\ &\quad +u'\left(w_2^*\right)\left(\frac{\partial y^*}{\partial c}(p-c)-y^*\right)\left[1-F(y^*)\right]-u\left(w_2^*\right)f(y^*)\frac{\partial y^*}{\partial c} \\ &= \int_0^{y^*} u'\left(w_1^*\right)\left(-y^*+(r-c)\frac{\partial y^*}{\partial c}\right)f(x)dx \\ &\quad +u'\left(w_2^*\right)\left(\frac{\partial y^*}{\partial c}(p-c)-y^*\right)\left[1-F(y^*)\right] \end{aligned}$$

The second term is negative, but the first has ambiguous sign, assuming that $\frac{\partial y^*}{\partial c} < 0$. A sufficient condition for the derivative to be negative is

$$(r-c)\frac{\partial y^*}{\partial c} < y^*$$

Notice that this can be expressed as

$$\eta \equiv -\frac{c}{y^*}\frac{\partial y^*}{\partial c} < \frac{c}{c-r}$$

That is, the elasticity of the optimal order with respect to the wholesale price, η, should be less than the ratio of the wholesale price to the difference between the wholesale price and the salvage value.

4 The supplier's optimal c

As we have already seen above, the newsboy's optimal demand for units of product is given by

$$y^*(c,r) \longleftarrow (r-c)\int_0^{y^*} u'(w_1^*)f(x)dx + u'(w_2^*)\,(p-c)\,[1-F(y^*)] = 0$$

The supplier choses c in order to maximise expected profit. We assume the supplier's marginal cost of units supplied is constant at q, and that units are only produced once they are ordered by the newsboy (so the supplier does not suffer any risk as regards the order that needs to be fulfilled). The supplier uses backward induction, taking into account the newsboy's optimal order $y^*(c,r)$, when setting the optimal price c. Therefore, the supplier has expected profit of

$$\begin{aligned}
E\pi_s(c) &= (c-q)y^* - r\int_0^{y^*}(y^*-x)f(x)dx \\
&= (c-q)y^* - ry^*\int_0^{y^*} f(x)dx + r\int_0^{y^*} xf(x)dx \\
&= (c-q)y^* - ry^*F(y^*) + r\left(y^*F(y^*) - \int_0^{y^*} F(x)dx\right) \\
&= (c-q)y^* - r\int_0^{y^*} F(x)dx
\end{aligned}$$

where at the second to last step$\int_0^{y^*} xf(x)dx$ was integrated by parts.

We now want to maximise $E\pi_s(c) = (c-q)y^* - r\int_0^{y^*} F(x)dx$ with respect to c. The first derivative of profit with respect to c is

$$\begin{aligned}
\frac{\partial E\pi_s(c)}{\partial c} &= y^* + (c-q)\frac{\partial y^*}{\partial c} - rF(y^*)\frac{\partial y^*}{\partial c} \\
&= y^* + \frac{\partial y^*}{\partial c}\left(c-q-rF(y^*)\right)
\end{aligned}$$

Under the assumption of DARA, $\frac{\partial y^*}{\partial c} < 0$, and so the sign of the first derivative is ambiguous if $c-q-rF(y^*) > 0$. The second derivative is

$$\frac{\partial^2 E\pi_s(c)}{\partial c^2} = \frac{\partial y^*}{\partial c} + \frac{\partial^2 y^*}{\partial c^2}\left(c-q-rF(y^*)\right) + \frac{\partial y^*}{\partial c}\left(1 - rf(y^*)\frac{\partial y^*}{\partial c}\right)$$

Assuming DARA and $c-q-rF(y^*) > 0$, a sufficient condition for expected profit to be concave in c is

$\frac{\partial^2 y^*}{\partial c^2} \leq 0$. We can also consider the the value of $\frac{\partial E\pi_s(c)}{\partial c}$ at $c = r$ and at $c = p$. In the first of these cases, the optimal order is $y^* = 1$, and we get

$$\left.\frac{\partial E\pi_s(c)}{\partial c}\right|_{c=r} = 1 - q\frac{\partial y^*}{\partial c} > 0$$

So it is always optimal for the supplier to increase c from its minimum. In the second case ($c = p$) the optimal order is $y^* = 0$, and we get

$$\left.\frac{\partial E\pi_s(c)}{\partial c}\right|_{c=r} = \frac{\partial y^*}{\partial c}(p - q) < 0$$

where the sign follows from the assumption that $p > q$, or else no profit would ever be possible. Thus it is always optimal for the supplier to decrease c from its maximum.

In short, this analysis of the extremes tells us that there must exist at least one strictly internal optimal value for c, i.e. a profit maximising supplier facing a given level of r will set c at some level that satisfies $r < c < p$. If we assume that indeed the supplier's expected profit is concave in c, then that internal optimum, denoted by c^*, is unique, and it satisfies

$$y^* + \frac{\partial y^*}{\partial c}(c^* - q - rF(y^*)) = 0 \tag{3}$$

where of course y^* is calculated with c^*.

Note that (3) can be written as

$$c^* = q + rF(y^*) - \frac{y^*}{\left(\frac{\partial y^*}{\partial c}\right)} \tag{4}$$

This is a standard mark-up equation, whereby the wholesale price is set at the marginal cost to the supplier, q, plus a strictly positive mark-up of $rF(y^*) - \frac{y^*}{\left(\frac{\partial y^*}{\partial c}\right)}$. Notice that, since the elasticity of the optimal order with respect to the wholesale price is defined as $\eta = -\frac{c}{y^*}\frac{\partial y^*}{\partial c}$, by multiplying top and bottom of the third term of the optimal pricing equation by c^*, we can see that it is equal to c divided by the elasticity. Therefore, the optimal price can be expressed as

$$c^* = q + rF(y^*) + \frac{c^*}{\eta}$$

So long as $\eta \neq 1$, this rearranges to the expression

$$c^* = \left(\frac{\eta}{\eta - 1}\right)(q + rF(y^*)) \tag{5}$$

Given this, in order for the optimal supply price to be positive (which by assumption it must be), the supplier must set the wholesale price such that $\eta > 1$. Equation (5) is a pricing arrangement that has both a summed mark-up over marginal supply cost q of $rF(y^*)$ which more than covers for the fact that the supplier must also insure, and a multiplicative mark-up factor that depends on the elasticity of the newsboy's demand with respect to the wholesale price.

We can split this pricing equation into two parts, one that is independent of the insurance arrangement,[9]

[9] In the sense that if there were no insurance at all, i.e. if $r = 0$, then the optimal supplier price would be set at $\left(\frac{\eta}{\eta-1}\right)q$. Of course, since η will in general depend upon r, there is still some (indirect) dependency of $\left(\frac{\eta}{\eta-1}\right)q$ upon r.

and so is the mark-up rule used for the sale of the unit to the newsboy, $\left(\frac{\eta}{\eta-1}\right)q$, and a second part that only refers to the insurance arrangement, $\left(\frac{\eta}{\eta-1}\right)rF(y^*)$. Since $\frac{\eta}{\eta-1} > 1$, the premium charged for the insurance arrangement is greater than $rF(y^*)$. On the other hand, the expected cost to the supplier of the insurance scheme (the salvage value of r on unsold items) is

$$r\int_0^{y^*}(y^*-x)f(x)dx$$

This is less than $rF(y^*)$, since $y^*-x<1$. Thus, the supplier charges an actuarially loaded (i.e. unfair) premium for the insurance arrangement. We should therefore expect that the optimal insurance demand by the newsboy is less than full coverage.

The comparative static question of interest is how c^* changes with r. We can investigate this by deriving the expression in (5) with respect to r:

$$\frac{\partial c^*}{\partial r} = \left(\frac{\partial}{\partial r}\left(\frac{\eta}{\eta-1}\right)\right)(q+rF(y^*)) + \left(\frac{\eta}{\eta-1}\right)\left(F(y^*)+rf(y^*)\frac{\partial y^*}{\partial r}\right)$$

Notice that the second term is positive, and the sign of the first term is the same as the sign of $\frac{\partial}{\partial r}\left(\frac{\eta}{\eta-1}\right)$. We can calculate

$$\begin{aligned}\frac{\partial}{\partial r}\left(\frac{\eta}{\eta-1}\right) &= \frac{\left(\frac{\partial\eta}{\partial r}\right)(\eta-1)-\eta\left(\frac{\partial\eta}{\partial r}\right)}{(\eta-1)^2} \\ &= -\frac{\frac{\partial\eta}{\partial r}}{(\eta-1)^2}\end{aligned}$$

which takes the opposite sign as $\frac{\partial\eta}{\partial r}$. Since $\eta = -\frac{c}{y^*}\frac{\partial y^*}{\partial c}$, we have

$$\frac{\partial\eta}{\partial r} = -\frac{\partial y^*}{\partial c}\left(-\frac{c}{y^{*2}}\frac{\partial y^*}{\partial r}\right) - \frac{c}{y^*}\frac{\partial^2 y^*}{\partial c\partial r}$$

The first term of this is negative (assuming DARA). The second term would also be negative if $\frac{\partial^2 y^*}{\partial c\partial r} > 0$. Thus, assuming that $\frac{\partial^2 y^*}{\partial c\partial r} > 0$, we can conclude that $\frac{\partial\eta}{\partial r} < 0$, which in turn implies that $\frac{\partial c^*}{\partial r} > 0$, the supplier's optimal price increases with the salvage value.

5 Insurance: the newsboy chooses r

We now consider the situation when the newsboy not only sets y^*, but also decides the level of r. If the newsboy sets $r=0$, then no insurance at all is demanded (the newsboy retains all risk), while if $r=c$ then full coverage is demanded (all risk is passed on to the supplier). Of course $0<r<c$ corresponds to partial coverage, or risk sharing between the newsboy and the supplier. With regards this problem, we already know that the per-unit price of coverage is set at a level that is strictly higher than what is actuarially fair. This should, in principle, imply that less than full coverage is always purchased, so long as the newsboy is strictly risk averse. It may also imply that no coverage at all is demanded. Intuitively, however, we might expect that the higher is risk aversion, the greater is the optimal level of insurance demand, that is, the optimal choice of r increases with risk aversion (at least beyond a threshold level of risk aversion, for which

the optimal r is actually positive).

The above discussion of the calculation of the supplier's optimal price is done for any given level of r, and so the price passed on to the newsboy can be expressed as $c^*(r)$. Given that, the newsboy's optimal demand for product is also only a function of r. Specifically, the optimal demand for any given r and c is denoted by $y^*(r,c)$, and so the demand for any given r and $c^*(r)$ is $y^*(r,c^*(r)) \equiv y^{**}(r)$. The effect of r upon the optimal order can now be seen to be

$$\frac{dy^{**}(r)}{dr} = \left.\frac{\partial y^*}{\partial r}\right|_c + \frac{\partial y^*}{\partial c}\frac{\partial c^*}{\partial r}$$

Note that this is now of indeterminate sign, since the first term is positive and the second is negative (under the assumptions made up to now).

We can also see that the newsboy's indirect utility function (calculated at $y^{**}(r)$) becomes only a function of r. The newsboy then must maximise this indirect utility function with an optimal choice of r. The assumptions and information that we have to study this problem are the following:

1. We assume that $\frac{\partial y^*}{\partial c} < 0$, which (for example) is the case under DARA.
2. We assume that $\frac{\partial c^*}{\partial r} > 0$, which (for example) is the case if $\frac{\partial^2 y^*}{\partial c \partial r} > 0$.

For any given r the newsboy's indirect expected utility function is

$$Ev(r) = \int_0^{y^{**}} u\left(w_1^{**}\right) f(x)dx + u\left(w_2^{**}\right)\left[1 - F(y^{**})\right]$$

where $w_1^{**} = x\left(p - c^*\right) + \left(y^{**} - x\right)\left(r - c^*\right)$ and $w_2^{**} = y^{**}\left(p - c^*\right)$. This is the function that we would like to maximise with respect to r.

The difficulty that we face in considering the newsboy's optimal choice of insurance is that unless we have exact functions, the maximisation is very much multi-leveled, and becomes so complex that a general analysis is impossible. For each choice of r, not only must we find the direct effect upon expected indirect utility, but also we must consider the indirect effect of r upon (i) the optimal level of product demand, and (ii) the optimal price charged by the supplier. And of course those two optimality considerations are themselves linked.

Given this complexity, rather than attempt a full theoretical investigation, we have preferred to look at concrete simulations.

6 Simulations

The entire problem has 4 "parameters", which are (1) q, the marginal cost of production of the supplier, (2) p, the retail price at which the newsboy sells to consumers, (3) f, the underlying probability density function for consumer demand, and (4) u, the newsboy's utility function. To carry out our simulations we have set (1) $q = 0.2$, (2) $p = 0.8$, (3) $f(x) = 1$ (i.e. uniform density), and (4) $u(x) = 2 - e^{-ax}$, where a is the newsboy's level of (constant) absolute risk aversion. We then work out the optimal level of r for a variety of different values of a.

The simulations were done using a computer programme that we wrote specifically for the problem that does the following. First, we construct a matrix of the optimal level of product demand, y, for given levels of r and c. Specifically, we use $c_1 = 0$, and we allow c to increase in discrete steps of size 0.001 from 0 up to $p - 0.001 = 0.799$.[10] For each c_j we consider values of r from 0 up to c_j, in steps of the same size as the steps taken for c. Then for each feasible pair (r_i, c_j) - that is, for each r_i, c_j such that $0 \le r_i \le c_j$, we use the newsboy's first-order condition for an optimal product order to calculate the optimal order quantity y_{ij}. Thus, we construct the following table:

$Y =$

	c_1	c_2	c_3	c_4	$\cdots$	c_n
r_1	y_{11}	y_{12}	y_{13}	y_{14}	$\cdots$	y_{1n}
r_2		y_{22}	y_{23}	y_{24}	$\cdots$	y_{2n}
r_3			y_{33}	y_{34}	$\cdots$	y_{3n}
r_4				y_{44}	$\cdots$	y_{4n}
$\vdots$					$\ddots$	$\vdots$
r_n					$\cdots$	y_{nn}

where $c_1 = r_1 = 0$, $c_j = 0.001(j-1)$, $r_i = 0.001(i-1)$, and $c_n = r_n = 0.799$.

Second, we use the same values of r and c and the supplier's expected profit function to calculate the level of supplier expected profit, given the corresponding y_{ij} value from table Y;

$\Pi =$

	c_1	c_2	c_3	c_4	$\cdots$	c_n
r_1	π_{11}	π_{12}	π_{13}	π_{14}	$\cdots$	π_{1n}
r_2		π_{22}	π_{23}	π_{24}	$\cdots$	π_{2n}
r_3			π_{33}	π_{34}	$\cdots$	π_{3n}
r_4				π_{44}	$\cdots$	π_{4n}
$\vdots$					$\ddots$	$\vdots$
r_n					$\cdots$	π_{nn}

Finally, the programme looks at each row r_i of table Π, and selects the largest value of π in that row, thereby identifying the optimal supplier price corresponding to that level of r; $c^*(r_i)$. Given that, we then go back to table Y, and for each row r_i we select the column corresponding to $c^*(r_i)$, in order to identify $y^{**}(r_i)$.

In this way, we now have, for each level of r, the optimal wholesale price that will be charged to the newsboy, $c^*(r)$, and the optimal order quantity $y^{**}(r)$, from which we can construct a vector containing the values of the newsboy's indirect expected utility;

$$V = [Ev(r_1), Ev(r_2), Ev(r_3), Ev(r_4), ..., Ev(r_n)]$$

The optimal choice of r is that which corresponds to the highest value in vector V.

Two comments are in order. First, because in general closed form solutions are not possible for this problem, we are forced to consider the values of c and r in discrete steps, rather than as continuous functions. This "discretisaton" of the problem of course leads to small errors in the final calculation of the newsboy's indirect expected utility. That is, not all of the values of $Ev(r)$ that we calculate in vector V do actually lie on the true continous function – some will, but others will lie close to but not actually on the function. However, so long as the steps taken during the process are very small, the errors made are also very small,

[10] Recall that using $r = c = p$ gives an indeterminate result, so we avoid using those numbers in our simulations.

and the vector of values V will lie closer and closer to the true continuous function.

Second, in the simulations we do not actually use the first-order condtion of the supplier to calculate the optimal wholesale price, but rather we calculate the supplier's expected profit numerically, and then select the value of c corresponding to the highest numerical value of expected profit. This is necessary, since the supplier's profit depends on c as well as on y. Solving the first-order condition would allow us to know the optimal c for any given y, but the levels of y to use in turn depend on c. That is, instead of matrix Π, we could envisage a matrix with r values for each row, and y values for each column, and then each matrix cell could be the level of $c(r, y)$ that satisfies the supplier's first-order condition. However, doing this would not allow us to identify which of these optimal value of c would actually be chosen for each r, since we would not know which of the combinations of c and y (for each r) actually gives the greatest level of supplier profit.

In the simulations, we set our step size for the calculation of our two matrices above at 0.001, that is, c goes from 0 to 0.799 in 800 consecutive steps. We allow the level of (constant) absolute risk aversion of the newsboy go from 1 (i.e. low level of risk aversion) to 5 (highly risk averse),[11] in steps of 1. Because of the discrete nature of the input data for the simulation, we end up with a discrete set of points for expected utility as a function of r that approximate the location of the true curve. Figure 1 shows the exact graphs of the sets of points generated from the simulations.

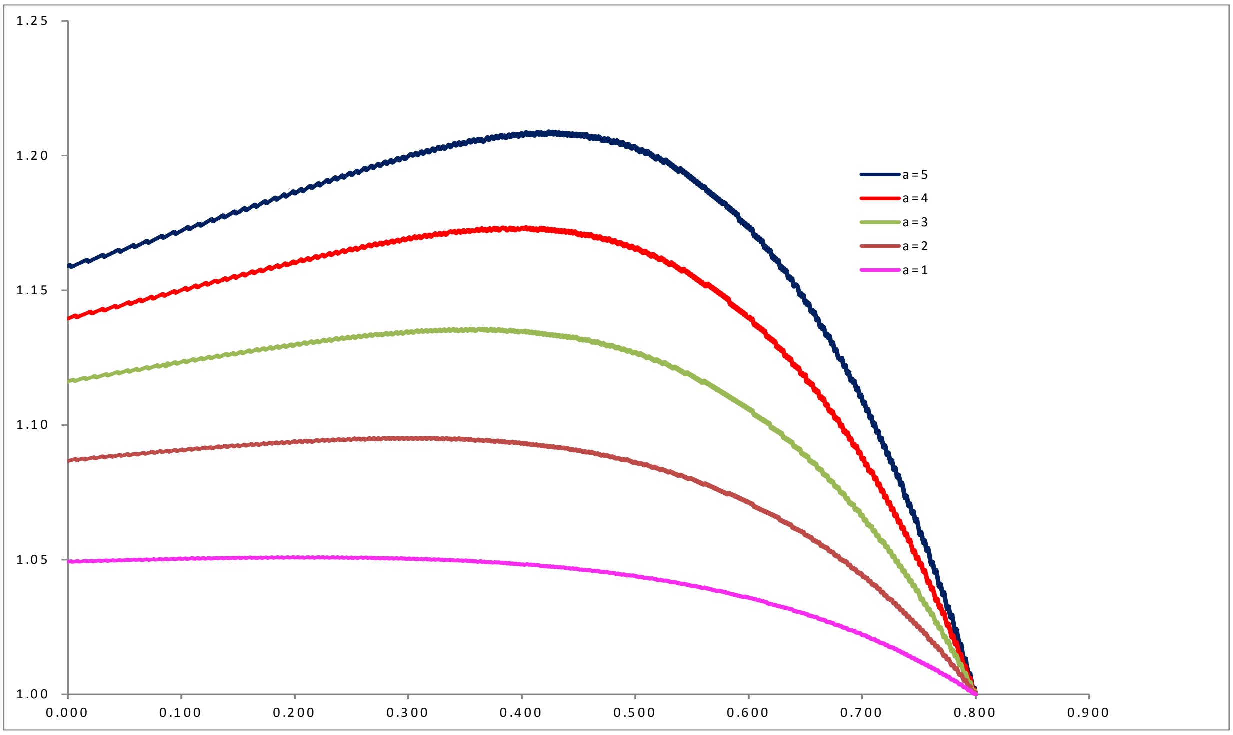

Figure 1: Raw simulation data

If we were to take a smaller step size, then the points would better approximate the true curve, but we would suffer from unacceptably long calculation time. Instead of doing that, we have simply run a

[11] We did not do the simulation with risk neutrality, for which the optimal r is 0 due to insurance carrying a loaded premium. It is also worthwhile noting that in fact absolute risk aversion of 5 may not be excessively high. Recall that our wealth variables are defined between 0 and 1, and since relative risk aversion is absolute risk aversion multiplied by wealth, even with absolute risk aversion of 5, we are considering cases of relative risk aversion that is less than, and possibly much less than, 5. By most empirical accounts, this is not an excessive degree of risk aversion.

least-square regression through the data that we calculate, and we take the resulting curve as our best approximation. We have restricted the regressions to being 4th order polinomials, and in all cases the fit is extemely high (all are well over $R^2 = 0.999$).[12] Table 1 reports the fitted curves, along with the maximum point of each of the regression curves, which is what we take as being our best estimate of the optimal value of r for each level of risk aversion. We note that the resulting indirect utility curves all have a strictly interior maximum, that is, in all cases $r^*(a) > 0$.

Table 1: Optimal insurance purchases by risk aversion, a

a	regression equation	R^2	r^*	$Eu(r^*)$
1	$-0.192r^4 + 0.112r^3 - 0.067r^2 + 0.0193r + 1.049$	0.9999	0.1978	1.0508
2	$-0.3416r^4 + 0.1205r^3 - 0.072r^2 + 0.0477r + 1.0866$	0.9999	0.3904	1.1728
3	$-0.4737r^4 + 0.0531r^3 - 0.0101r^2 + 0.072r + 1.1161$	0.9999	0.3552	1.1352
4	$-0.6246r^4 - 0.0298r^3 + 0.093r^2 + 0.0997r + 1.1399$	0.9998	0.3904	1.1728
5	$-0.8131r^4 - 0.0928r^3 + 0.2199r^2 + 0.0996r + 1.1596$	0.9998	0.4162	1.2081

As can be seen in figure 2, all of the maximum points lie on a curve (shown in the figure as a dashed curve) with strictly positive slope,[13] which indicates that the optimal level of insurance, r^*, in our simulation increases with risk aversion.

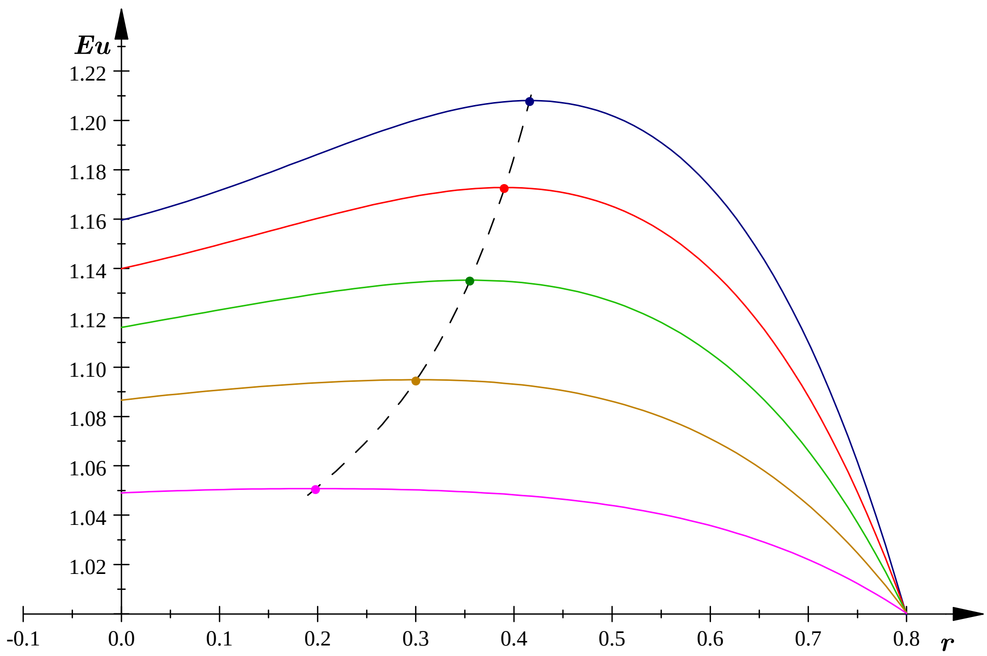

Figure 2: Indirect utility as a function of insurance, r, for different levels of risk aversion, a. Magenta is $a = 1$, brown is $a = 2$, green is $a = 3$, red is $a = 4$, and blue is $a = 5$.

Finally, we also note that in all cases the optimal insurance demand is for partial insurance, that is $r^* < c(r^*)$. Of course this is entirely to be expected since we know that the premium charged for insurance

[12] There is no effect at all upon the fit of taking higher order polinomials.

[13] The exact equation for the curve displayed is $13.248r^4 - 7.4866r^3 + 1.8212r^2 + 0.0814r + 1.0011$, which fits the five data points with $R^2 = 1$. It is strictly increasing on $0.19 \leq r \leq 0.43$, which encompasses all of the five points in question.

at any r is actuarially loaded. In figure 3 we show the graphs of the optimal wholesale price for each level of risk aversion as a function of the demand for insurance r.[14] The dashed line is the line where $c = r$. For all possible levels of insurance demand r, and for all of the levels of risk aversion that we study, the optimal wholesale price is always greater than the insurance variable. Thus, the optimal level of insurance for each level of risk aversion also satisfies partial coverage.

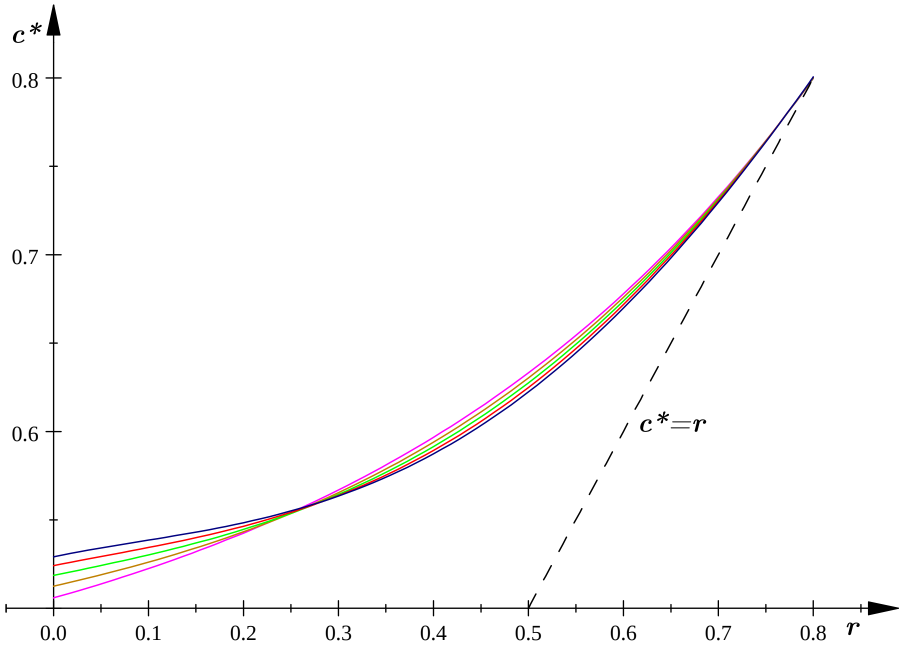

Figure 3: $c^*(r)$: $a = 1$ in magenta, $a = 2$ in brown, $a = 3$ in green, $a = 4$ in red, and $a = 5$ in blue.

7 Conclusion

In this paper we have considered optimal insurance in the standard newsboy problem, with a weakly risk averse newsboy. The insurance mechanism used is simply the salvage value at which excess stock can be sold back to the supplier. The main additions to the existing literature are (i) to allow the supplier to set the wholesale price optimally, taking into account the salvage value, and (ii) to allow the newsboy to choose the salvage value as an insurance demand.

We find that the optimal wholsale price can be expressed in terms of a mark-up equation, that depends (among other things) on the elasticity of the newsboy's demand to the wholesale price. The wholesale price will always be set such that this elasticity is valued above 1. Second, the supplier charges, implicitly, a premium for the insurance mechanism that is actuarially unfair to the newsboy.

In regards the insurance demand, we are only able to study this problem using a numerical simulation, since the complexity of the problem does not allow closed form solutions to be found in general. The simulation was carried out under an assumption of the newsboy having constant (and positive) absolute risk aversion, and that the demand for the newsboy's sales is uniformly distributed. Under those conditions, we

[14] Here we only show the 4^{th} order approximations to the sets of points in question. The R^2 values for each fitted curve are all again over 0.999.

find that (i) for strictly positive levels of risk aversion, the newsboy sets the salvage value at a strictly positive number, which indicates a strictly positive demand for insurance, (ii) for all risk aversion cases considered, insurance demand is partial (i.e less than full coverage is demanded), and (ii) the level of insurance demanded grows with risk aversion.

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