Prudence, temperance (and other virtues): The dual story*

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Abstract

We characterize m^{th} order dual stochastic dominance in Yaari's [32] dual theory by deriving simple nested classes of lottery pairs such that the direction of preference between these lotteries is equivalent to signing the m^{th} derivative of the probability weighting (distortion) function, with m a positive integer. To reach this goal, we explicate the appropriate dual story to stand on equal footing with that to interpret the signs of the successive derivatives of the utility function in expected utility. In this way, we also obtain a simple and intuitive interpretation of the well-known fact that expected utility and the dual theory diverge from the 3^{rd} order onwards.

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1 Introduction

Although first received with some skepticism, the notions of prudence and temperance have now been widely accepted almost on par with the fundamental concept of risk aversion, at least in an expected utility (E.U.) framework.

The expanding use of these notions, sometimes termed "higher order risk attitudes", can be explained by the fact that they were progressively given a more general interpretation within the E.U. model. Consider prudence for instance. The term was coined by Kimball [21] in an influential paper in which he showed that precautionary savings is characterized in an E.U. framework by a positive 3^{rd} derivative of the utility function (i.e., $U''' \geq 0$ or "prudence"). However, it is by now well-known that this positive sign of U''' can be justified more generally outside the specific decision problem of saving. This alternative justification of prudence was initiated by Menezes, Geiss and Tressler [23], who used the term "downside risk aversion", and it was further pursued in Eeckhoudt and Schlesinger [14], who also showed how to proceed from prudence to higher order risk attitudes. These authors first state a "model free" preference, namely that decision makers (D.M.'s) like to "combine good with bad" instead of having to face either everything good or everything bad.[1] Next, this model free preference is shown to be translated into prudence ($U''' \geq 0$) within the E.U. model, and from prudence —by defining a sequence of nested lotteries and always supplying the preference for combining good with bad— the higher order risk attitudes may be obtained similarly, starting with temperance at the 4^{th} order.

It turns out besides that the simple model free interpretation of prudence and higher order risk attitudes found in Eeckhoudt and Schlesinger [14] lends itself easily to experimental verification. As a result, there is now an intensive experimental research activity around the concepts of prudence and temperance in an E.U. framework (e.g., Ebert and Wiesen [9], Deck and Schlesinger [7] and Noussair, Trautmann and van de Kuilen [25], to name a few).

While prudence, temperance and possibly higher order risk attitudes can be initially presented as natural properties in a model free environment, their interpretation has been developed so far exclusively within an E.U. framework.[2] Our purpose here is to explicate the appropriate story for the dual theory (D.T.) of choice under risk (Yaari [32]). Specifically, we consider the question of how to derive simple nested classes of lottery pairs such that the direction of preference between these lotteries is equivalent to signing the m^{th} derivative of the probability weighting (or distortion) function in Yaari's dual theory, with m an arbitrary positive integer. It turns out that this requires a fundamental departure from the approach of Eeckhoudt and Schlesinger [14], which, as we will show, is unable to deliver the desired implications in the D.T. framework.

One may therefore say that, in some sense, this paper constitutes the analog of Eeckhoudt and Schlesinger [14] for Yaari's [32] dual theory: the contribution of the present paper to the extant literature on (dual or inverse) stochastic ordering in D.T. (Muliere and Scarsini [24], Wang and Young [33] and Chateauneuf, Gajdos and Wilthien [2]) is similar to the contribution

[1] This idea was already present in Eeckhoudt and Schlesinger [14] but the phrase "combine good with bad" formally appeared in Eeckhoudt, Schlesinger and Tsetlin [15]. Notice that other names have been given for this attitude. For instance Chiu [4] refers to a "precedence relation" in which good precedes bad, while Eeckhoudt, Rey and Schlesinger [13] speak of "correlation aversion" in the spirit of contributions by Richard [27] and Epstein and Tanny [18].

[2] In a very recent work, Baillon [1] generalizes these interpretations of prudence and higher order risk attitudes to a setting featuring ambiguity.

of the paper by Eeckhoudt and Schlesinger [14] to the extant literature on (primal) stochastic ordering in E.U. (Whitmore [34], Menezes, Geiss and Tressler [23] and, in particular, Ekern [17]). But while the dual story retains generic features of the primal story (e.g., a precedence relation) it crucially departs from it in its implementation (e.g., by reference to the so-called dual moments and to squeezing). As such, this paper represents a first step towards a more general interpretation of higher order risk attitudes, covering alternative decision models, such as rank-dependent utility and prospect theory (Kahneman and Tversky [20], Quiggin [26], Schmeidler [29, 30], Tversky and Kahneman [31]), for which D.T. is a building block. Uncovering the (eventually simple) details of the dual story required a subtle and delicate analysis.

As is well-known, primal and dual stochastic dominance coincide up to the 2^{nd} order and diverge from the 3^{rd} order onwards. As a by-product, which is of interest in its own right, the model free story appropriate for D.T. will nicely make apparent the fundamental reason behind this divergence.

To reach our goal, our paper is organized as follows. In Section 2, we fix the notation and setting, and introduce some preliminaries for the D.T. decision model. In Section 3, we provide some basic intuition behind our results, by presenting simple numerical illustrations that make the links between the model free assumption and the two models of choice under risk (E.U. versus D.T.) explicit. Section 4 discusses the construction of higher order risk attitudes within D.T. by assuming appropriate model free preferences. Section 5 illustrates implications of our results for optimal portfolio choice. Section 6 shows that the sign of the 3^{rd} derivative of the probability weighting function is naturally linked to a self-protection problem. Section 7 is devoted to the formal proofs of the results illustrated in Section 3 and of the construction explicated in Section 4. We conclude in Section 8 with a summary of the results and an indication of potential extensions.

2 Notation and Setting

We represent an n-state lottery A, assigning probabilities p_i to outcomes x_i, $i = 1, \ldots, n$, by $A = [\ x_1, p_1\ ;\ \ldots\ ;\ x_n, p_n\ ]$. With each lottery, generating a probability distribution over outcomes, one can associate a random variable. Henceforth, the lottery and its associated random variable are often identified. Furthermore, we write $\mathrm{P}[A \leq x] = F_A(x) = 1 - S_A(x)$. We always assume that states are ordered according to their associated outcomes, from the lowest outcome state (1) to the highest outcome state (n). Outcomes are assumed to be non-negative.

Under Yaari's [32] dual theory, the value V of an n-state lottery A, with $0 \leq x_1 \leq \cdots \leq x_n$, is given by

$$\begin{aligned} \mathrm{V}[A] &= \int_0^\infty x \, \mathrm{d}h(F_A(x)) \\ &= \sum_{i=1}^{n} x_i [h(F_A(x_i)) - h(F_A(x_{i-1}))], \end{aligned}$$

with $x_0 = 0$ and $F_A(x_0) = 0$ by convention, and $h : [0,1] \to [0,1]$ with $h(0) = 0$, $h(1) = 1$, $h' \geq 0$, a probability weighting (distortion) function, henceforth assumed to satisfy $h'' \leq 0$ (strong risk aversion) and to be differentiable for all degrees of differentiation on $(0,1)$. We sometimes denote by $h^{(m)}$ the m^{th} derivative of h.[3] Furthermore, we denote by $\succeq$ the (weak) preference relation induced by V or, in some cases, by its E.U. counterpart.

We say that a lottery B precedes lottery A in 3[rd] dual (or inverse) stochastic order if

$$\mathrm{E}[A] \le \mathrm{E}[B], \qquad \mathrm{E}[\min(A_1, A_2)] \le \mathrm{E}[\min(B_1, B_2)], \quad \text{and} \quad {}^3S_{-A}^{-1} \ge {}^3S_{-B}^{-1}.$$

Here, A_1 (B_1) and A_2 (B_2) are two independent draws from lottery A (B), ${}^{m+1}S_A^{-1}(q) = \int_0^q {}^mS_A^{-1}(p)\,\mathrm{d}p$, $m = 1, 2, \ldots$, $0 \le q \le 1$, and inequalities between functions are understood pointwise; see e.g., Muliere and Scarsini [24] and Wang and Young [33]. We refer to $\mathrm{E}[\min(A_1, A_2, \ldots, A_m)]$ as the m^{th} dual moment of A.[4] Similarly, we say that lottery B precedes lottery A in 4[th] dual stochastic order if

$$\mathrm{E}[A] \le \mathrm{E}[B], \qquad \mathrm{E}[\min(A_1, A_2)] \le \mathrm{E}[\min(B_1, B_2)],$$
$$\mathrm{E}[\min(A_1, A_2, A_3)] \le \mathrm{E}[\min(B_1, B_2, B_3)], \quad \text{and} \quad {}^4S_{-A}^{-1} \ge {}^4S_{-B}^{-1}.$$

In full generality, we say that lottery B precedes lottery A in m^{th} dual stochastic order if

$$\mathrm{E}[A] \le \mathrm{E}[B], \quad \mathrm{E}[\min(A_1, A_2)] \le \mathrm{E}[\min(B_1, B_2)], \quad \ldots \quad,$$
$$\mathrm{E}[\min(A_1, A_2, \ldots, A_{m-1})] \le \mathrm{E}[\min(B_1, B_2, \ldots, B_{m-1})], \quad \text{and} \quad {}^mS_{-A}^{-1} \ge {}^mS_{-B}^{-1}.$$

For further details on dual (inverse) stochastic ordering, we refer to De La Cal and Cárcamo [6] and the references therein.

3 Intuition and Illustration

Since it is well-known that at the 1[st] and 2[nd] orders E.U. and D.T. coincide in their evaluation of a sure reduction in wealth and of a mean preserving spread (see Chew, Karni and Safra [3] and Roëll [28]), we develop here intuition and illustrations at the 3[rd] and higher orders.

To motivate their paper in which they interpret the sign of U''', Menezes, Geiss and Tressler [23] first refer to Mao's lotteries[5] given by

$$A = [\ 0, 1/4\ ;\ 2, 3/4\ ], \qquad B = [\ 1, 3/4\ ;\ 3, 1/4\ ].$$

[3] We use the notations h', h'', ... and $h^{(1)}$, $h^{(2)}$, ... interchangeably.

[4] In statistics, these moments are sometimes referred to as mean (first) order statistics. They measure the expected worst outcome in an experiment with repeated independent draws. Primal moments occur under E.U. when considering power functions as utility functions. Dual moments occur under D.T. when considering power functions as probability weighting functions:

$$\begin{aligned}
\mathrm{E}[\min(A_1, A_2, \ldots, A_m)] &= \int_0^\infty (1 - F_A(x))^m\,\mathrm{d}x \\
&= \int_0^\infty \bar{h}(1 - F_A(x))\,\mathrm{d}x \\
&= \int_0^\infty x\,\mathrm{d}\left(1 - \bar{h}(1 - F_A(x))\right) \\
&= \int_0^\infty x\,\mathrm{d}h(F_A(x)) \\
&= \mathrm{V}[A],
\end{aligned}$$

with $\bar{h}(p) = p^m$ and $\bar{h}(p) = 1 - h(1-p)$, $0 \le p \le 1$. Notice that $\bar{h} : [0,1] \to [0,1]$ with $\bar{h}(0) = 0$, $\bar{h}(1) = 1$, $\bar{h}' \ge 0$ and $\bar{h}'' \ge 0$.

[5] This name originates from the fact that these lotteries were used by James Mao [22] in an experiment with business men.

Notice that A and B have the first two moments in common, that is, have the same mean and variance, and economic agents who like to "combine good with bad" prefer B to A. Under E.U., this corresponds to $U''' \geq 0$ (see e.g., Eeckhoudt and Schlesinger [14] for a precise statement of this result).

Lotteries A and B can be (viewed as) generated from a common initial lottery L given by

$$L = [\ 1, 1/2\ ;\ 2, 1/2\ ],$$

to which a zero-mean lottery ε given by

$$\varepsilon = [\ -1, 1/2\ ;\ 1, 1/2\ ],$$

is allocated. If the zero-mean lottery is allocated to the worst outcome of L (i.e., 1) one generates A:

$$A = [\ 1 + \varepsilon, 1/2\ ;\ 2 + 0, 1/2\ ],$$

while if the zero-mean lottery is allocated to the best outcome of L (i.e., 2) one generates B:

$$B = [\ 1 + 0, 1/2\ ;\ 2 + \varepsilon, 1/2\ ].$$

One notices that B is obtained from L by allocating the bad outcome ε (bad since ε is second order stochastically dominated by 0) to the good outcome of L while the reverse is true for A. One might alternatively say[6] that in A the bad (ε) precedes the good (0). In B, on the contrary, the good (0) precedes the bad (ε).

Now consider D.T. instead of E.U. as decision model. What do we know for D.T.? Notice first that A and B also have the first two dual moments in common, where the second dual moments amount to:

$$\mathrm{E}\left[\min(A_1, A_2)\right] = (1 - (3/4)^2)0 + (3/4)^2 2 = 9/8.$$
$$\mathrm{E}\left[\min(B_1, B_2)\right] = (1 - (1/4)^2)1 + (1/4)^2 3 = 9/8.$$

Furthermore, one easily computes that

$$\mathrm{V}[A] = h(1/4)0 + (1 - h(1/4))2, \qquad \mathrm{V}[B] = h(3/4)1 + (1 - h(3/4))3,$$

with $h(0) = 0$, $h(1) = 1$, $h' \geq 0$, $h'' \leq 0$. We now have the following result:

Proposition 3.1 *If h is quadratic,* $\mathrm{V}[A] = \mathrm{V}[B]$.

Proof. Let $h(p) = \alpha p - \beta p^2$. Note that $h(0) = 0$. To have $h(1) = 1$, $\alpha - \beta = 1$ should hold. Hence, $h(p) = (1 + \beta)p - \beta p^2$, so that $h'(p) = 1 + \beta - 2\beta p$. To have $h'(1) \geq 0$ (hence $h'(p) \geq 0$ whenever $h''(p) \leq 0$), $\beta \leq 1$ should hold. Furthermore, $h''(p) = -2\beta$ so $h'' \leq 0$ if $\beta \geq 0$. In sum, $0 \leq \beta \leq 1$ and $\alpha = 1 + \beta$. With such a probability weighting function,

$$\mathrm{V}[A] = ((1/4)(1 + \beta) - (1/16)\beta)0 + (1 - (1/4)(1 + \beta) + (1/16)\beta)2 = 3/2 - (3/8)\beta.$$
$$\mathrm{V}[B] = ((3/4)(1 + \beta) - (9/16)\beta)1 + (1 - (3/4)(1 + \beta) + (9/16)\beta)3 = 3/2 - (3/8)\beta.$$

□

[6] This alternative presentation will be useful to understand the dual story explicated in the next section.

Proposition 3.1 says that, under D.T., the agent is indifferent between Mao's lotteries if the distortion function is quadratic. It suggests that, for this specific pair of lotteries, preferences are dictated by the behavior of higher-order (> 2) derivatives of the distortion function. This is indeed confirmed by the next result (the proof of which is deferred to Section 7 —see Theorem 7.2 and Appendix 1):

Proposition 3.2 *If* $h''' \geq (\leq)\ 0$, $\mathrm{V}[A] \leq (\geq)\ \mathrm{V}[B]$, *that is, B is preferred (dispreferred) to A.*

Since E.U. and D.T. coincide in their evaluations of Mao's lotteries at the 3[rd] order (under E.U. $U''' \geq 0 \Rightarrow B \succeq A$ and under D.T. $h''' \geq 0 \Rightarrow B \succeq A$), one may wonder whether this result extends to all lottery pairs defined by Eeckhoudt and Schlesinger [14] at the 3[rd] order (see, specifically, Definition 1 on p. 282 of Eeckhoudt and Schlesinger [14]). By providing two numerical examples, we show that E.U. and D.T. may, but need not, coincide for such Eeckhoudt and Schlesinger pairs of lotteries.[7]

First, consider an initial lottery $\tilde{L}$ given by

$$\tilde{L} = [\ 4, 1/2\ ;\ 10, 1/2\ ],$$

and then allocate, as in Eeckhoudt and Schlesinger [14], to one of the outcomes the zero-mean lottery ζ given by

$$\zeta = [\ -2, 1/2\ ;\ 2, 1/2\ ].$$

This generates either $\tilde{A}$ or $\tilde{B}$ given by

$$\tilde{A} = [\ 2, 1/4\ ;\ 6, 1/4\ ;\ 10, 1/2\ ], \qquad \tilde{B} = [\ 4, 1/2\ ;\ 8, 1/4\ ;\ 12, 1/4\ ]. \tag{3.1}$$

Clearly, $\tilde{A}$ and $\tilde{B}$ again have the first two moments in common. It is well-known that under E.U.,

$$U''' \geq (\leq)\ 0 \Rightarrow \tilde{B} \succeq (\preceq) \tilde{A}.$$

Now let's consider the comparison between $\tilde{A}$ and $\tilde{B}$ under D.T. One easily verifies that $\tilde{A}$ and $\tilde{B}$ have the same mean and the same dual moments $\mathrm{E}\left[\min(\tilde{A}_1, \tilde{A}_2)\right]$ and $\mathrm{E}\left[\min(\tilde{B}_1, \tilde{B}_2)\right]$, which amount to

$$\mathrm{E}\left[\min(\tilde{A}_1, \tilde{A}_2)\right] = (7/16)2 + (5/16)6 + (4/16)10 = 21/4.$$
$$\mathrm{E}\left[\min(\tilde{B}_1, \tilde{B}_2)\right] = (12/16)4 + (3/16)8 + (1/16)12 = 21/4.$$

We have the following result:

Proposition 3.3 *If h is quadratic,* $\mathrm{V}[\tilde{A}] = \mathrm{V}[\tilde{B}]$.

Proof. With a probability weighting function that satisfies the same properties as in the proof of Proposition 3.1,

$$\begin{aligned}
\mathrm{V}[\tilde{A}] =& ((1/4)(1+\beta) - (1/16)\beta)2 + (((1/2)(1+\beta) - (1/4)\beta) - ((1/4)(1+\beta) - (1/16)\beta))6 \\
&+ (1 - ((1/2)(1+\beta) - (1/4)\beta))10 = 7 - (7/4)\beta. \\
\mathrm{V}[\tilde{B}] =& ((1/2)(1+\beta) - (1/4)\beta)4 + (((3/4)(1+\beta) - (9/16)\beta) - ((1/2)(1+\beta) - (1/4)\beta))8 \\
&+ (1 - ((3/4)(1+\beta) - (9/16)\beta))12 = 7 - (7/4)\beta.
\end{aligned}$$

[7] It was sometimes conjectured —falsely, as we know now— that the coincidence of E.U. and D.T. at the 3[rd] order is true only for Mao's lotteries. Therefore we have included the explicit additional example (3.1).

□

Besides, following the proofs developed in Section 7 (see Theorem 7.2 and Appendix 1), we find that:

Proposition 3.4 *If* $h''' \geq (\leq)\ 0$, $\mathrm{V}[\tilde{A}] \leq (\geq)\ \mathrm{V}[\tilde{B}]$, *that is,* $\tilde{B}$ *is preferred (dispreferred) to* $\tilde{A}$.

It thus turns out that for some lotteries used by Eeckhoudt and Schlesinger [14] at the 3$^{\text{rd}}$ order, E.U. and D.T. produce the same ranking at the 3$^{\text{rd}}$ order. However, we now show by means of another example that this is not always the case. This "counterexample" serves to undermine the false impression given by Mao's and, in fact, various other lotteries in the class of lotteries used by Eeckhoudt and Schlesinger [14] at the 3$^{\text{rd}}$ order, such as (3.1), that E.U. and D.T. coincide at the 3$^{\text{rd}}$ order.[8]

Indeed, start from an initial lottery $\hat{L}$ given by

$$\hat{L} = [\ 1, 1/2\ ;\ 2, 1/2\ ],$$

and allocate to one of the outcomes the zero-mean lottery θ given by

$$\theta = [\ -2, 1/3\ ;\ 1, 2/3\ ].$$

Notice that this zero-mean risk θ is very different from the previous ones we used (ε, ζ): it is no longer symmetric and besides (and more importantly) it will induce a change in ranking of the outcomes. Indeed, the new lotteries, after the apportionment of θ, are $\hat{A}$ and $\hat{B}$ given by

$$\hat{A} = [\ -1, 1/6\ ;\ 2, 5/6\ ], \qquad \hat{B} = [\ 0, 1/6\ ;\ 1, 3/6\ ;\ 3, 2/6\ ].$$

From Eeckhoudt and Schlesinger [14], we know that under E.U.,

$$U''' \geq (\leq)\ 0 \Rightarrow \hat{B} \succeq (\preceq)\ \hat{A},$$

and notice that $\mathrm{E}[\hat{A}] = \mathrm{E}[\hat{B}]$, $\mathrm{Var}[\hat{A}] = \mathrm{Var}[\hat{B}]$, while $\mathrm{Skew}[\hat{A}] < \mathrm{Skew}[\hat{B}]$.

If we turn to D.T. and consider a quadratic distortion function so that $h''' = 0$, it can be shown that $\hat{A} \succeq \hat{B}$, thus producing a ranking at the 3$^{\text{rd}}$ order different from E.U.: under E.U. $\hat{A}$ and $\hat{B}$ are equivalent whenever $U''' = 0$. Indeed, with a probability weighting function that satisfies the same properties as in the proof of Proposition 3.1,

$$\begin{aligned}
\mathrm{V}[\hat{A}] =& -((1/6)(1+\beta) - (1/36)\beta)1 + (1 - ((1/6)(1+\beta) - (1/36)\beta))2 \\
&= 3/2 - (5/12)\beta. \\
\mathrm{V}[\hat{B}] =& ((1/6)(1+\beta) - (1/36)\beta)0 + (((4/6)(1+\beta) - (16/36)\beta) - ((1/6)(1+\beta) - (1/36)\beta))1 \\
&\quad + (1 - ((4/6)(1+\beta) - (16/36)\beta))3 \\
&= = 3/2 - (7/12)\beta.
\end{aligned}$$

Hence, $\mathrm{V}[\hat{A}] - \mathrm{V}[\hat{B}] = (1/6)\beta \geq 0$, since $0 \leq \beta \leq 1$, with strict inequality whenever $\beta > 0$ (or $h'' > 0$). The basic reason for this result is the fact that the second dual moments of $\hat{A}$ and $\hat{B}$ differ. In the next section, we provide a detailed discussion of the fundamental reason behind the divergence between E.U. and D.T. from the 3$^{\text{rd}}$ order onwards.

[8] This divergence is, in principle, well-known but shown here explicitly for the Eeckhoudt and Schlesinger [14] construction of apportioning a zero-mean risk.

We have seen that, at the 3^{rd} order, the E.U. and D.T. decision models treat some pairs of lotteries equally, but do not coincide in their evaluation of other pairs of lotteries. How about the 4^{th} order? In Appendix 2, we illustrate that E.U. and D.T. do not agree in their evaluation of a specific pair of lotteries used by Eeckhoudt and Schlesinger [14] at the 4^{th} order (although they may agree in other cases, just as at the 3^{rd} order —something we leave to the reader to verify).

It thus appears that the model free prescription favoring the "combination of good with bad" in the particular way as suggested by Eeckhoudt and Schlesinger [14] naturally leads to an interpretation of the signs of the successive derivatives of U in the E.U. model; however, this same model free principle does not always generate a similar interpretation for the signs of the successive derivatives of the probability weighting function under the D.T. framework.

As a result, if one wants to obtain for D.T. and the signs of the successive derivatives of the probability weighting function h a development that parallels that within E.U., one has to modify the initial model free preferences. This is the purpose of the next section.

4 Model Free Preferences for the Dual Theory

In order to give an intuitive interpretation of the signs of the successive derivatives of the probability weighting function, we now specify the appropriate model free preferences. We will continue to consider that, under D.T., D.M.'s satisfy a precedence relation so as to favor "good preceding bad", as for the E.U. model. However, our definition of good and bad will now be based on the concepts of "squeezing" and "anti-squeezing" a distribution (Eeckhoudt and Hansen [10]), instead of using zero-mean risks compared to zero with certainty, as in Eeckhoudt and Schlesinger [14].

To illustrate, consider a lottery $C^{(2)}$ given by[9]

$$C^{(2)} = [\ 1, 1/2\ ;\ 3, 1/2\ ].$$

Now transform the lottery $C^{(2)}$ to a lottery $D^{(2)}$ by squeezing, that is, by subtracting $1/M$ $(M \geq 1)$[10] from 3 and moving it to 1. Such a squeeze is good (under $h'' \leq 0$) and it yields the new lottery $D^{(2)}$ given by

$$D^{(2)} = [\ 1 + 1/M, 1/2\ ;\ 3 - 1/M, 1/2\ ],$$

with $D^{(2)} \succeq C^{(2)}$. Of course, an anti-squeezing of $C^{(2)}$ that is equivalent (but opposite) to the squeezing would produce $E^{(2)}$ given by

$$E^{(2)} = [\ 1 - 1/M, 1/2\ ;\ 3 + 1/M, 1/2\ ],$$

with $C^{(2)} \succeq E^{(2)}$. At the 2^{nd} order, these preferences correspond to $h'' \leq 0$ under D.T. and to $U'' \leq 0$ under the E.U. model. This is true since anti-squeezing and squeezing are special cases of a mean preserving spread and a mean preserving contraction, and we know that D.T. and E.U. coincide in their evaluation of a mean preserving spread and a mean preserving contraction.

[9]The superscript $^{(2)}$ refers to "2^{nd} order" and more generally the superscript $^{(m)}$ refers to m^{th} order.
[10]The condition on M guarantees that the squeeze does not change the initial ranking of outcomes.

Turning now to the 3rd order, the agreement between D.T. and E.U. may collapse, as already suggested in Section 3. The model free preferences based on the precedence relation towards good and bad coupled with the notions of squeezing and anti-squeezing yield:

$$\text{Start from } C^{(3)} = [\ 1, 1/4\ ;\ 3, 1/4\ ;\ 5, 1/4\ ;\ 7, 1/4\ ],$$

then generate $D^{(3)}$ as $D^{(3)} = [\ 1 + 1/M, 1/4\ ;\ 3 - 1/M, 1/4\ ;\ 5 - 1/M, 1/4\ ;\ 7 + 1/M, 1/4\ ]$. In fact, to obtain $D^{(3)}$ from $C^{(3)}$:

(i) One first works on the worst outcomes of $C^{(3)}$ (1 and 3) and one applies to them a squeeze (which is good), subtracting $1/M$ from 3 and moving it to 1.

(ii) Then on the best outcomes of $C^{(3)}$ (5 and 7), one does an equivalent anti-squeezing (which is bad), moving outcome from the worst state (5) to the best state (7).

In the transition from $C^{(3)}$ to $D^{(3)}$, the good precedes the bad, and as proved in the next section (to which a precise statement of this result is deferred —see Theorems 7.2 and 7.4):

$$h''' \geq (\leq)\ 0 \Leftrightarrow D^{(3)} \succeq (\preceq)\ C^{(3)},$$

while $D^{(3)} \sim C^{(3)}$ induces $h''' = 0$.

At this stage, it is very important (crucial) to stress that for E.U. maximizers with $U''' \geq 0$, there is no unanimity for the comparison between $C^{(3)}$ and $D^{(3)}$, essentially because these two lotteries with the same mean have different variances and skewnesses. Indeed, one has $\mathrm{Var}[D^{(3)}] > \mathrm{Var}[C^{(3)}]$ and $\mathrm{Skew}[D^{(3)}] > \mathrm{Skew}[C^{(3)}]$. As a result, E.U. D.M.'s who are relatively more (less) risk averse than prudent prefer $C^{(3)}$ ($D^{(3)}$).[11]

While our story of squeezing and anti-squeezing yields different primal moments (variances in particular) for the lotteries that are compared, it is worth observing that it preserves equality of the second dual moments. Indeed one may verify that

$$\mathrm{E}\left[\min(D_1^{(3)}, D_2^{(3)})\right] = \mathrm{E}\left[\min(C_1^{(3)}, C_2^{(3)})\right].$$

Our combination of squeezing and anti-squeezing affects the variance while leaving the second dual moment unchanged, which is the fundamental reason why E.U. and D.T. diverge from the 3rd order onwards.

In order to interpret now the sign of the 4th derivative of h we start from an expanded lottery $C^{(4)}$ given by

$$C^{(4)} = [\ 1, 1/8\ ;\ 3, 1/8\ ;\ 5, 1/8\ ;\ 7, 1/8\ ;\ 9, 1/8\ ;\ 11, 1/8\ ;\ 13, 1/8\ ;\ 15, 1/8\ ].$$

Now on the first four outcomes of $C^{(4)}$ (i.e., from 1 to 7) which are "bad" relative to the other four outcomes, we do the beneficial transformation described for the 3rd order. Then for the four best outcomes of $C^{(4)}$ (i.e., from 9 to 15) we do exactly the opposite 3rd order transformation (which is bad) so that we obtain lottery $D^{(4)}$ given by

$$D^{(4)} = [\ 1 + 1/M, 1/8\ ;\ 3 - 1/M, 1/8\ ;\ 5 - 1/M, 1/8\ ;\ 7 + 1/M, 1/8\ ;$$
$$9 - 1/M, 1/8\ ;\ 11 + 1/M, 1/8\ ;\ 13 + 1/M, 1/8\ ;\ 15 - 1/M, 1/8\ ],$$

[11] While they disagree in many cases, the dual story may yield the same result as the primal story of Eeckhoudt and Schlesinger [14] at the 3rd order in specific cases. We illustrate this explicitly for Mao's lotteries and the Eeckhoudt and Schlesinger [14] pair of lotteries (3.1) in Appendix 1.

where all outcomes are equally likely (probability=1/8) and where —as before— $M \geq 1$.

In the next section, we prove that such transformations are unanimously appreciated (rejected) under D.T. by D.M.'s with $h'''' \leq 0$ ($h'''' \geq 0$), while indifference prevails when $h'''' = 0$ (see Theorems 7.7 and 7.9).

In order to see the implications of the dual story in the E.U. model, it is convenient to compare $C^{(4)}$ with the $\hat{D}^{(4)}$ lottery obtained from $D^{(4)}$ when $M = 1$, i.e.,

$$\hat{D}^{(4)} = [\ 2, 2/8\ ;\ 4, 1/8\ ;\ 8, 2/8\ ;\ 12, 1/8\ ;\ 14, 2/8\ ].$$

While $C^{(4)}$ and $\hat{D}^{(4)}$ have the same mean and skewness (the latter is equal to zero) it appears that $\hat{D}^{(4)}$ has a larger variance and a lower kurtosis. Hence, there cannot be unanimity among the risk averse and temperate E.U. D.M.'s about the appreciation of the two lotteries: some will prefer $\hat{D}^{(4)}$ while others will prefer $C^{(4)}$, depending upon their relative degrees of risk aversion and temperance. This observation makes explicit the reason why at the 4^{th} order E.U. and D.T. (continue to) diverge: while the sequence of squeezing and anti-squeezing at the 4^{th} order produces different primal moments for $C^{(4)}$ and $D^{(4)}$, it preserves the equality of the third dual moments, corroborating again their relevance for the dual story.

Since the lotteries at the 4^{th} order are generated by a sequence of lotteries relevant at the 3^{rd} order, this analysis can be pursued up to any order to interpret the sign of the equivalent (same order) derivatives of the probability weighting function. We thus obtain, as in the Eeckhoudt and Schlesinger [14] approach, a sequence of simple nested lotteries that yield now the appropriate interpretation of the sign of $h^{(m)}$.

5 Portfolio Choice with Derivatives

Let us now consider a D.T. investor with initial sure wealth w_0. Suppose that he allocates an amount α to a risky asset (the "stock") and an amount $w_0 - \alpha$ to a risk-free asset (the "bond"). The bond (stock) earns a sure (risky) return of r (R) per unit invested. We assume that r and R are independent of the amounts invested. The investor's problem is to determine the optimal amount α^*. Because $(w_0 - \alpha)(1 + r) + \alpha(1 + R) = w_0(1 + r) + \alpha(R - r)$, his problem reads:

$$\arg\max_{\alpha}\{\alpha(\mathrm{V}[R] - r)\}. \tag{5.1}$$

We add the constraint $0 \leq \alpha \leq w_0$. It readily follows that the optimal solution is a corner solution.

Suppose that $\mathrm{V}[R] < r$. Then the optimal solution is $\alpha^* = 0$: the investor fully invests in the bond. Now imagine that the investor is offered an improvement in the m^{th} dual order sense à la Ekern to the distribution of the risky return, with $m \geq 2$. In particular, consider the possibility of supplementing the stock with derivative products on the stock, having zero expected value and at zero cost. The derivative products, which can be used to manage the risk generated by the stock, are selected (see below for details) such that they lead to an m^{th} dual order improvement of R; the risky portfolio of stock *and* derivatives is denoted by $\hat{R}$.[12] Then two cases are possible: (i) If $\mathrm{V}[\hat{R}] \leq r$, full investment in the bond remains optimal. (ii) If the improvement is sufficiently large so that $\mathrm{V}[\hat{R}] > r$, the D.T. investor will shift from

[12] The analysis that follows is somewhat connected to the analysis of Dittmar [8] and of Crainich, Eeckhoudt and Trannoy [5], both in the E.U. model.

one corner solution (full investment in the bond i.e., $\alpha^* = 0$) to the other corner solution (full investment in the risky portfolio i.e., $\alpha^* = w_0$).

Hence, we find that an m^{th} dual order improvement of R never reduces the demand for the risky asset. This stands in sharp contrast to the E.U. theory, where an m^{th} primal order improvement of R has ambiguous effects, even for $m \geq 1$. To obtain the natural result that the improvement induces a higher demand of the risky asset under E.U., restrictions need to be imposed on the structure of the problem (see Section 9.3 of Eeckhoudt and Gollier [11] and Gollier [19] for details).

To illustrate the m^{th} dual order improvements using derivatives, assume throughout w.l.o.g. that $r = 0$. Consider the 2^{nd} order first. Assume that, similar to the previous section, the stock price takes the values 1 and 3 with probability 1/2. It follows from our story in the previous section that a long position in a put option (which is good) combined with a short position in a call option (which is bad), each with a strike price of 2 such that the joint expected value is zero, improves the attractiveness of the risky portfolio ($\hat{R}$ versus R) whenever $h'' \leq 0$. We note that the same result would be true under E.U. with h replaced by U. As visualized in the first panel of Figure 1, this combination of long put and short call provides a hedge against adverse stock scenarios, which is financed by giving up some upward potential.

Next, consider the 3^{rd} order. Assume that, similar to the previous section, the stock price takes the values 1, 3, 5 and 7 with probability 1/4. Then a so-called "straddle" option at stock price 4 such that the expected value is zero, improves the attractiveness of the risky portfolio ($\hat{R}$ versus R) whenever $h''' \geq 0$. We emphasize that, because the straddle increases not only the skewness but also the variance of the risky portfolio, under E.U., contrary to under D.T., there is no unanimity for the straddle supplement. As visualized in the second panel of Figure 1, the straddle pays off in bad stock scenarios and in good stock scenarios, but generates losses in intermediate stock scenarios.

Finally and similarly, at the 4^{th} order, assume that the stock price takes the values 1, 3, 5, 7, 9, 11, 13 and 15 with probability 1/8. A long straddle at stock price 4 combined with a short straddle at stock price 12, with joint expected value equal zero, improves the attractiveness of the risky portfolio whenever $h'''' \leq 0$. This combination of long and short straddle[13], visualized in the last panel of Figure 1, is a popular and simple case of a so-called "volatility spread". We notice that, upon supplementing the stock with derivative products on the stock, the ranking of outcomes is always preserved.

6 Self-Protection with Background Risk

Next, consider a D.T. agent with initial sure wealth w_0 facing the risk of losing the random amount $L > 0$. The probability of occurrence of the loss, p, depends on the self-protection[14] effort, e, exerted by the agent: $p(e)$ is decreasing in e, $0 \leq p(e) \leq 1$. Effort is measured in monetary equivalents. The agent's problem is to determine the optimal level of effort e^* maximizing his D.T. value.

We will analyze this problem in the presence of an independent background risk. This background risk may e.g., be due to the risk from holding risky financial assets or to uncertain labor income. It will appear that the sign of h''' plays an essential role in this self-protection

[13] The payoffs of the long and short straddle are (digitally) set to 0 at stock price 8 for simplicity.

[14] Adopting the terminology of Ehrlich and Becker [16].

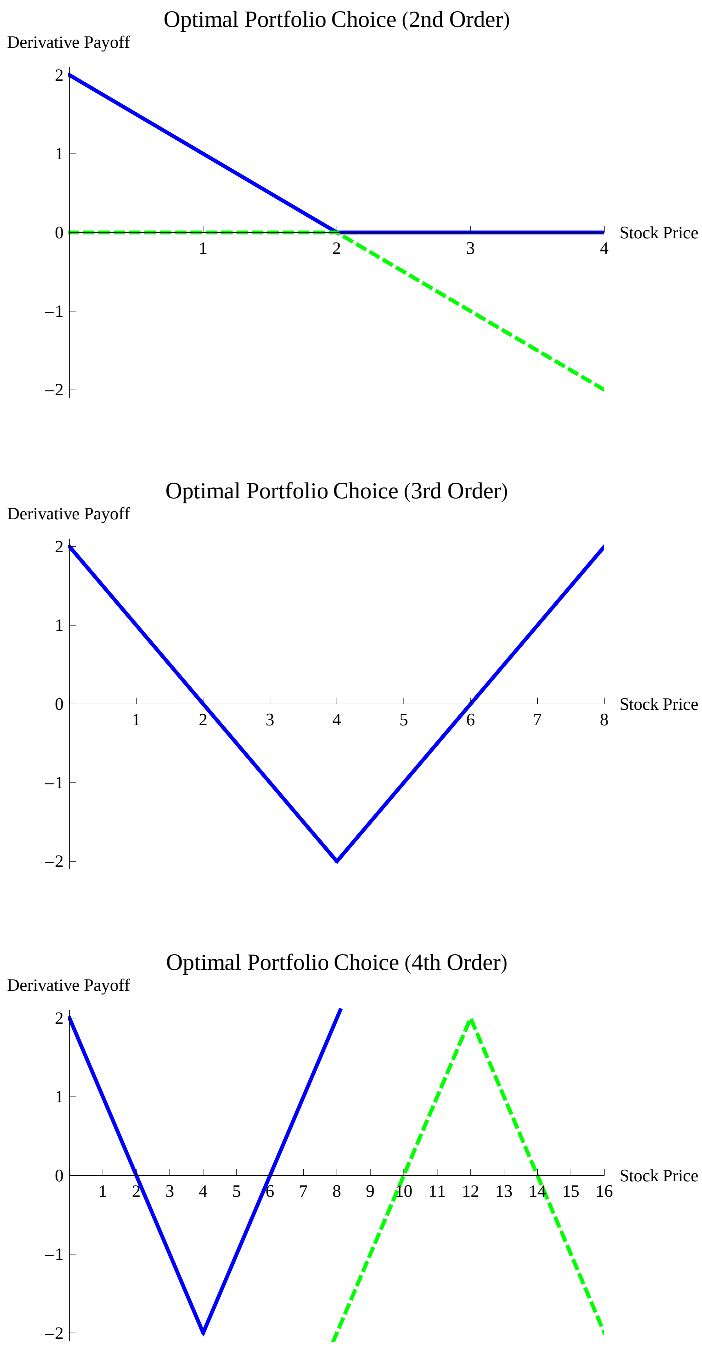

Figure 1: Optimal Portfolio Choice with Derivatives.
This figure plots the payoff functions of the derivative products used to supplement the stock and manage the risk it generates. At the 2^{nd} order, we consider a long position in a put option (blue) and a short position in a call option (green, dashed), both with a strike price of 2. At the 3^{rd} order, we consider a straddle option (blue) at stock price 4. Finally, at the 4^{th} order, we consider a long straddle at stock price 4 (blue) and a short straddle at stock price 12 (green, dashed). At each order, the derivative products have zero (joint) expected value.

(or, prevention) problem with background risk. Hence, under E.U., the a priori validity of $U''' \geq 0$ is obtained from a savings decision (Kimball [21]). For D.T., we show that the a priori validity of $h''' \geq 0$ is linked to a self-protection problem.

With an independent binary background risk, taking the value $\pm\varepsilon$ with probability 1/2, $\varepsilon > 0$, the self-protection problem can be represented by the following 4-state lottery:

$$S = [\, w_0 - L - \varepsilon - e, p(e)/2 \,;\; w_0 - L + \varepsilon - e, p(e)/2 \,;\; w_0 - \varepsilon - e, (1-p(e))/2 \,;\; w_0 + \varepsilon - e, (1-p(e))/2 \,],$$

assuming $2\varepsilon < L$. In the complementary case that $2\varepsilon > L$, the middle two states of S change places (because states are ordered according to their associated outcomes). The agent's problem reads:

$$\arg\max_e \{\mathrm{V}\,[S]\}. \tag{6.1}$$

We assume throughout that V is concave in e. We assume in particular that p is twice differentiable and satisfies $p''(e) \geq 0$.

If $2\varepsilon < L$,

$$\begin{aligned}\mathrm{V}\,[S] =& h(p(e)/2)(w_0 - L - \varepsilon - e) + (h(p(e)) - h(p(e)/2))(w_0 - L + \varepsilon - e)\\ &+ (h((1 + p(e))/2) - h(p(e)))(w_0 - \varepsilon - e) + (1 - h((1 + p(e))/2))(w_0 + \varepsilon - e),\end{aligned}$$

and the first-order condition for optimality reads

$$\begin{aligned}\frac{d\mathrm{V}\,[S]}{de} =& (p'(e)/2)h'(p(e)/2)\,((w_0 - L - \varepsilon - e) - (w_0 - L + \varepsilon - e))\\ &+ p'(e)h'(p(e))\,((w_0 - L + \varepsilon - e) - (w_0 - \varepsilon - e))\\ &+ (p'(e)/2)h'((1 + p(e))/2)\,((w_0 - \varepsilon - e) - (w_0 + \varepsilon - e))\\ &- 1 = 0.\end{aligned}$$

After obvious simplifications, we obtain

$$\begin{aligned}p'(e)\varepsilon\left(-h'(p(e)/2) + 2h'(p(e)) - h'((1 + p(e))/2)\right)&\\ - p'(e)h'(p(e))L - 1 = 0.&\end{aligned} \tag{6.2}$$

Note first that in the absence of any background risk, i.e., $\varepsilon = 0$, the first-order condition reduces to

$$-p'(e)h'(p(e))L - 1 = 0; \tag{6.3}$$

cf. Eeckhoudt and Gollier [12]. Now assume, as in Eeckhoudt and Gollier [12] for the E.U. model, that $-p'(e)h'(p(e))L - 1 = 0$ when $p(e) = 1/2$. This means that the D.T. D.M. optimally selects an effort level e such that the probability of the occurrence of a loss is 1/2, in the absence of background risk. Under this assumption, we find from (6.2) that the impact of the background risk is linked to the sign of

$$-h'(1/4) + 2h'(1/2) - h'(3/4), \tag{6.4}$$

which depends on the sign of h'''.

Next, consider the case $2\varepsilon > L$. In this case, the agent's problem reads (6.1) with

$$\begin{aligned}\mathrm{V}\,[S] =& h(p(e)/2)(w_0 - L - \varepsilon - e) + (h(1/2) - h(p(e)/2))(w_0 - \varepsilon - e)\\ &+ (h((1 + p(e))/2) - h(1/2))(w_0 - L + \varepsilon - e) + (1 - h((1 + p(e)/2)))(w_0 + \varepsilon - e),\end{aligned}$$

and the first-order condition is

$$\begin{aligned}\frac{d\mathrm{V}\,[S]}{de} =& (p'(e)/2)h'(p(e)/2)\left((w_0 - L - \varepsilon - e) - (w_0 - \varepsilon - e)\right)\\ &+ (p'(e)/2)h'((1+p(e))/2)\left((w_0 - L + \varepsilon - e) - (w_0 + \varepsilon - e)\right)\\ &- 1 = 0,\end{aligned}$$

which, after obvious simplifications, reduces to

$$-(1/2)p'(e)L\left(h'(p(e)/2) + h'((1+p(e))/2)\right) - 1 = 0. \tag{6.5}$$

Recall that in the absence of any background risk, the first-order condition equals (6.3). Hence, noting that

$$\begin{aligned}&\left(-(1/2)p'(e)L\left(h'(p(e)/2) + h'((1+p(e))/2)\right) - 1\right) - \left(-p'(e)h'(p(e))L - 1\right)\\ &= -(1/2)p'(e)L\left(h'(p(e)/2) - 2h'(p(e)) + h'((1+p(e))/2)\right),\end{aligned} \tag{6.6}$$

and maintaining the assumption that $-p'(e)h'(p(e))L - 1 = 0$ when $p(e) = 1/2$, we find that the impact of the background risk is again linked to the sign of (6.4) which, in turn, depends on the sign of h'''.

So, under the assumption that $-p'(e)h'(p(e))L - 1 = 0$ when $p(e) = 1/2$, $h''' \geq 0$ guarantees that the marginal benefit of self-protection is increased and the background risk stimulates self-protection.

Notice that there is a difference between the two cases: When $2\varepsilon < L$, the convexity of h' adds a marginal benefit that depends on the size of ε, when compared to the case of no background risk; cf. (6.2). When $2\varepsilon > L$, the convexity of h' adds a marginal benefit that is independent of ε but depends on the size of L, when compared to the case of no background risk; cf. (6.6).

Now suppose $-p'(e)h'(p(e))L - 1 = 0$ when $p(e) \neq 1/2$. In particular, suppose $2\varepsilon < L$, and $-p'(e)h'(p(e))L - 1 = 0$ when $p(e) < 1/2$. The introduction of the background risk is then linked to the sign of

$$\begin{aligned}&\left(-h'(p(e)/2) + 2h'(p(e)) - h'((1+p(e))/2)\right)\\ &= \left([h'(p(e)) - h'(p(e)/2)] - [h'((1+p(e))/2) - h'(1/2)] - [h'(1/2) - h'(p(e))]\right).\end{aligned}$$

Under concavity of h', $[h'(p(e)) - h'(p(e)/2)] - [h'((1+p(e))/2) - h'(1/2)] \geq 0$. Furthermore, under concavity of h, $-[h'(1/2) - h'(p(e))] \geq 0$. So, if $-p'(e)h'(p(e))L - 1 = 0$ when $p(e) < 1/2$, concavity of h and h' leads (when multiplied by $p'(e)\varepsilon$) to a negative impact in the first-order condition (6.2), hence a reduction of effort, in view of the concavity of V.

Next, with $2\varepsilon < L$, suppose $-p'(e)h'(p(e))L - 1 = 0$ when $p(e) > 1/2$. The introduction of the background risk is then linked to the sign of

$$\begin{aligned}&\left(-h'(p(e)/2) + 2h'(p(e)) - h'((1+p(e))/2)\right)\\ &= \left([h'(1/2) - h'(p(e)/2)] - [h'((1+p(e))/2) - h'(p(e))] - [h'(1/2) - h'(p(e))]\right).\end{aligned}$$

Under convexity of h', $[h'(1/2) - h'(p(e)/2)] - [h'((1+p(e))/2) - h'(p(e))] \leq 0$. Furthermore, under concavity of h, $-[h'(1/2) - h'(p(e))] \leq 0$. So, if $-p'(e)h'(p(e))L - 1 = 0$ when $p(e) > 1/2$, concavity of h and convexity of h' lead to a positive impact in the first-order condition (6.2), hence an increase in effort. The case $2\varepsilon > L$ follows similarly, in view of (6.6).

7 Theorems and Proofs

Throughout this section, the n states of a lottery have equal probability $(1/n)$ of occurring. But the increments in the outcomes, when moving to a higher state, are permitted to equal zero (as long as the squeezes and anti-squeezes we perform below do not change the initial ranking of outcomes), effectively yielding lotteries with unequal state probabilities. Recall that, for an n-state lottery A with each outcome $x_1 \leq x_2 \leq \cdots \leq x_n$ having equal probability $1/n$, the second and third dual moments are given by

$$\mathrm{E}\left[\min(A_1, A_2)\right] = (1/n^2)x_n + (3/n^2)x_{n-1} + (5/n^2)x_{n-2} + (7/n^2)x_{n-3} + \cdots + ((2n-1)/n^2)x_1,$$
$$\mathrm{E}\left[\min(A_1, A_2, A_3)\right] = (1/n^3)x_n + (7/n^3)x_{n-1} + (19/n^3)x_{n-2} + (37/n^3)x_{n-3} + \cdots + ((3(n-1)n+1)/n^3)x_1.$$

7.1 The 3^{rd} Order

We start by deriving a simple class of lottery pairs such that the direction of preference between these lotteries is equivalent to signing the 3^{rd} derivative of the probability weighting function. Consider, with $\delta, \delta' > 0$,

$$G^{(3)} = [\ \delta, 1/n\ ;\ \ldots;\ -\delta, 1/n\ ] \quad \text{and} \quad B^{(3)} = [\ -\delta', 1/n\ ;\ \ldots;\ \delta', 1/n\ ].$$

The acronyms G and B refer to "good" and "bad". Upon adding $G^{(3)}$ and $B^{(3)}$ state-wise to an arbitrarily given initial lottery (having at least three states, such that the state-wise addition is feasible), we generate $D^{(3)}$ if $G^{(3)}$ state-wise precedes $B^{(3)}$ and generate $C^{(3)}$ if $B^{(3)}$ state-wise precedes $G^{(3)}$. We note that: (i) δ and δ' need not be equal; (ii) the number of states that "…" denotes in $G^{(3)}$ and $B^{(3)}$ may differ; (iii) the number of states that "…" denotes in $G^{(3)}$ and $B^{(3)}$ is non-negative and may equal zero; (iv) we do not require equal spacing between the two blocks $G^{(3)}$ and $B^{(3)}$ vs. $B^{(3)}$ and $G^{(3)}$ when adding them to the initial lottery and may, in fact, also have negative spacing, so that the two blocks overlap (but with $G^{(3)}$ strictly preceding $B^{(3)}$ for $D^{(3)}$ and $B^{(3)}$ strictly preceding $G^{(3)}$ for $C^{(3)}$). All as long as the initial ranking of outcomes remains unaffected and the outcomes of the resulting lotteries remain non-negative.

Remark 7.1 *We note that the δ's in $G^{(3)}$ and $B^{(3)}$ need not sum to zero. That is, we do not need $+\delta, +\delta'$ and $-\delta, -\delta'$. We could have, for example, $+\delta_1, +\delta'_1$ and $+\delta_2, +\delta'_2$, where $\delta_1 > \delta_2$ and $\delta'_1 > \delta'_2$. However, for simplicity, we assume in the analysis, without losing generality, that they do sum to zero. This is for expositional ease only.*

The reasoning is rather trivial. We don't compare the changed distribution to the original. Rather we compare two altered distributions: one has the good precede the bad, the other the reverse. If the δ's sum to 3 instead of zero, both alterations will share the same mean — which of course is 3 higher than the mean of the original distribution. This shift in the mean is irrelevant, since we do not compare anything to the original distribution.

Theorem 7.2 *If $C^{(3)}$ and $D^{(3)}$ are according to the above construction, then, $D^{(3)}$ is preferred to $C^{(3)}$ by any D.T. D.M. having $h''' \geq 0$.*

Proof. Recall that the probabilities of generating the minimal outcome in a two-shot experiment (two independent draws) for the states of an n-state lottery are

$$(2n-1)/n^2, \ldots, 7/n^2, 5/n^2, 3/n^2, 1/n^2,$$

respectively. Hence, it follows that

$$\mathrm{E}\left[C^{(3)}\right] = \mathrm{E}\left[D^{(3)}\right],$$
$$\mathrm{E}\left[\min(C_1^{(3)}, C_2^{(3)})\right] = \mathrm{E}\left[\min(D_1^{(3)}, D_2^{(3)})\right],$$

and $S_{-C^{(3)}}$ surpasses $S_{-D^{(3)}}$ after crossing twice, so that $D^{(3)}$ precedes $C^{(3)}$ in 3[rd] dual stochastic order; see e.g., Proposition 4.9 of Wang and Young [33]. Indeed, adding $G^{(3)}$ and $B^{(3)}$ with $G^{(3)}$ state-wise preceding $B^{(3)}$ and adding $B^{(3)}$ and $G^{(3)}$ with $B^{(3)}$ state-wise preceding $G^{(3)}$ has the same impact on the dual moments up to the 2[nd] order. □

Now let's consider special subclasses of $C^{(3)}$ and $D^{(3)}$, sufficient for our purpose of signing h'''. Arbitrarily fix (n, x) with $n \geq 3, x \geq 0$ and consider the lottery

$$\begin{aligned} C_{n,x}^{(3)} =&[\ 0, 1/n\ ;\ \ldots;\ x-1, 1/n\ ;\\ &\mathbf{x}, \mathbf{1/n}\ ;\ \mathbf{x+1}, \mathbf{1/n}\ ;\ \mathbf{x+2}, \mathbf{1/n}\ ;\\ &x+3, 1/n;\ \ldots\ ;\ ., 1/n\ ]. \end{aligned}$$

The three states in bold need to be present for any $n(\geq 3)$, the remaining states are added until the state probabilities sum up to 1.

The increments in the outcomes when moving to a higher state for the lottery class $C_{n,x}^{(3)}$ are normalized to be 1, but may be generalized to be arbitrarily non-negative, as long as the squeezes and anti-squeezes we perform below do not change the initial ranking of outcomes.[15] We notice in particular that increments equal to zero may be permitted, effectively yielding lotteries with unequal state probabilities.

Let $M \geq 3$. Then, adding (state-wise) to the three states in bold

$$C2D_{n,x}^{(3)} = [\ 1/M, 1/n\ ;\ -2/M, 1/n\ ;\ 1/M, 1/n\ ]$$

yields $D_{n,x}^{(3)}$:

$$\begin{aligned} D_{n,x}^{(3)} =&[\ 0, 1/n\ ;\ \ldots;\ x-1, 1/n\ ;\\ &\mathbf{x+1/M}, \mathbf{1/n}\ ;\ \mathbf{x+1-2/M}, \mathbf{1/n}\ ;\ \mathbf{x+2+1/M}, \mathbf{1/n}\ ;\\ &x+3, 1/n;\ \ldots\ ;\ ., 1/n\ ]. \end{aligned}$$

Lottery $D_{n,x}^{(3)}$ is obtained from lottery $C_{n,x}^{(3)}$ by squeezing the distribution at the lowest two states in bold and equivalently anti-squeezing the distribution at the highest two states in bold. To keep the required number of states as small as possible, to enhance parsimony of our approach, we now squeeze and anti-squeeze the distribution at overlapping states, different from Section 4.

Note that $C_{n,x}^{(3)}$ and $D_{n,x}^{(3)}$ occur as subclasses of $C^{(3)}$ and $D^{(3)}$ by taking, for example,

$$G^{(3)} = [\ 1/(2M), 1/n\ ;\ -1/(2M), 1/n\ ] \quad \text{and} \quad B^{(3)} = [\ -1/(2M), 1/n\ ;\ 1/(2M), 1/n\ ],$$

[15] This is accomplished by restricting M in the squeezes and anti-squeezes explicated below accordingly.

and adding them state-wise to the bold states of an initial lottery $L^{(3)}$ given by

$$L^{(3)} = [\ 0, 1/n\ ;\ \dots;\ x-1, 1/n\ ; \\ \mathbf{x+1/(2M), 1/n}\ ;\ \mathbf{x+1-1/M, 1/n}\ ;\ \mathbf{x+2+1/(2M), 1/n}\ ; \\ x+3, 1/n;\ \dots\ ;\ ., 1/n\ ].$$

$D^{(3)}_{n,x}$ occurs if $G^{(3)}$ precedes (by one state) $B^{(3)}$ while $C^{(3)}_{n,x}$ occurs if $B^{(3)}$ precedes (by one state) $G^{(3)}$.

Corollary 7.3 *If $C^{(3)}_{n,x}$ and $D^{(3)}_{n,x}$ are according to the above construction, then, for any $n \geq 3, x \geq 0$, $D^{(3)}_{n,x}$ is preferred to $C^{(3)}_{n,x}$ by any D.T. D.M. having $h''' \geq 0$.*

Proof. The result follows from Theorem 7.2 in view of the fact that $C^{(3)}_{n,x}$ and $D^{(3)}_{n,x}$ are subclasses of $C^{(3)}$ and $D^{(3)}$. □

Theorem 7.4 *If, for any $n \geq 3, x \geq 0$, lottery $D^{(3)}_{n,x}$ is preferred to lottery $C^{(3)}_{n,x}$ by a D.T. D.M., then $h''' \geq 0$.*

Proof. Since h'' is differentiable on $(0,1)$, the third derivative of h being positive means (is equivalent to requiring) that

$$([h(p_+) - h(p_0)] - [h(p_0) - h(p_-)]) - ([h(p_0) - h(p_-)] - [h(p_-) - h(p_{--})]) \\ = h(p_+) - 3h(p_0) + 3h(p_-) - h(p_{--}) \geq 0,$$

for any four equidistant points $0 \leq p_{--} \leq p_- \leq p_0 \leq p_+ \leq 1$. We will show that this is satisfied whenever lottery $D^{(3)}_{n,x}$ is preferred to lottery $C^{(3)}_{n,x}$ for any $n \geq 3, x \geq 0$.

Notice that the D.T. value of a lottery A with n outcomes $x_0 = 0 \leq x_1 \leq \cdots \leq x_n$ is given by

$$\begin{aligned} \mathrm{V}[A] &= \int_0^\infty x \,\mathrm{d}\left(1 - \bar{h}(1 - F_A(x))\right) \\ &= -\sum_{i=1}^n x_i[\bar{h}(1 - F_A(x_i)) - \bar{h}(1 - F_A(x_{i-1}))] \\ &= \int_0^\infty \bar{h}(1 - F_A(x))\,\mathrm{d}x \\ &= \sum_{i=1}^n \bar{h}(1 - F_A(x_{i-1}))(x_i - x_{i-1}), \end{aligned}$$

with $\bar{h}(p) = 1 - h(1-p)$. Notice that $\bar{h} : [0,1] \to [0,1]$ with $\bar{h}(0) = 0$, $\bar{h}(1) = 1$, $\bar{h}' \geq 0$ and $\bar{h}'' \geq 0$. Here, $F_A(x_0) = 0$ by convention, so $1 - F_A(x_0) = \bar{h}(1 - F_A(x_0)) = 1$. Hence,

$$\begin{aligned} \mathrm{V}[C^{(3)}_{n,x}] &= \sum_{i=1}^{j} \bar{h}(1 - F_{C^{(3)}_{n,x}}(x_{i-1}))(x_i - x_{i-1}) + \sum_{i=j+1}^{j+4} \bar{h}(1 - F_{C^{(3)}_{n,x}}(x_{i-1}))(x_i - x_{i-1}) \\ &\quad + \sum_{i=j+5}^{n} \bar{h}(1 - F_{C^{(3)}_{n,x}}(x_{i-1}))(x_i - x_{i-1}) \\ &=: \Sigma_1^{C^{(3)}_{n,x}} + \Sigma_2^{C^{(3)}_{n,x}} + \Sigma_3^{C^{(3)}_{n,x}}, \end{aligned}$$

with j corresponding to the state preceding the three states in bold that are always present, and similarly for $D^{(3)}_{n,x}$. Notice that

$$\Sigma_1^{C^{(3)}_{n,x}} = \Sigma_1^{D^{(3)}_{n,x}}, \qquad \Sigma_3^{C^{(3)}_{n,x}} = \Sigma_3^{D^{(3)}_{n,x}}.$$

Furthermore,

$$\Sigma_2^{C^{(3)}_{n,x}} = 1\bar{h}(1-F(x_j)) + 1\bar{h}(1-F(x_{j+1})) + 1\bar{h}(1-F(x_{j+2})) + 1\bar{h}(1-F(x_{j+3})),$$

and

$$\begin{aligned}\Sigma_2^{D^{(3)}_{n,x}} =&(1+1/M)\bar{h}(1-F(x_j)) + (1-3/M)\bar{h}(1-F(x_{j+1})) \\ &+ (1+3/M)\bar{h}(1-F(x_{j+2})) + (1-1/M)\bar{h}(1-F(x_{j+3})),\end{aligned}$$

suppressing the indices $C^{(3)}_{n,x}$ and $D^{(3)}_{n,x}$ for convenience. Define, for given $n \geq 3, x \geq 0$, the probability q, $0 \leq q \leq 1-3/n$, such that

$$\begin{aligned} F(x_j) = q, \qquad & F(x_{j+1}) = q+1/n, \qquad F(x_{j+2}) = q+2/n, \\ & F(x_{j+3}) = q+3/n, \qquad F(x_{j+4}) = q+4/n. \end{aligned}$$

Then

$$\Sigma_2^{C^{(3)}_{n,x}} =\bar{h}(1-q) + \bar{h}(1-q-1/n) + \bar{h}(1-q-2/n) + \bar{h}(1-q-3/n)$$

and

$$\begin{aligned}\Sigma_2^{D^{(3)}_{n,x}} =&(1+1/M)\bar{h}(1-q) + (1-3/M)\bar{h}(1-q-1/n) \\ &+ (1+3/M)\bar{h}(1-q-2/n) + (1-1/M)\bar{h}(1-q-3/n).\end{aligned}$$

Defining $p_{--}, p_{-}, p_0, p_{+}$ such that

$$\begin{aligned} 3/n \leq 1-q = p_+ \leq 1, \qquad & 1-q-1/n = p_0, \\ 1-q-2/n = p_-, \qquad & 1-q-3/n = p_{--}, \end{aligned}$$

it then follows from the arbitrariness of $n \geq 3, x \geq 0$, hence the arbitrariness of $0 \leq q \leq 1-3/n$, that

$$\Sigma_2^{D^{(3)}_{n,x}} - \Sigma_2^{C^{(3)}_{n,x}} = \bar{h}(p_+) - 3\bar{h}(p_0) + 3\bar{h}(p_-) - \bar{h}(p_{--}) \geq 0,$$

for any four equidistant points $0 \leq p_{--} \leq p_- \leq p_0 \leq p_+ \leq 1$, whenever $D^{(3)}_{n,x}$ is preferred to $C^{(3)}_{n,x}$ for any $n \geq 3, x \geq 0$. Finally, notice that $\bar{h}''' \geq 0$ is equivalent to $h''' \geq 0$. □

Remark 7.5 If $h''' = 0$, then $C^{(3)}_{n,x}$ and $D^{(3)}_{n,x}$ are equivalent: $C^{(3)}_{n,x}$ and $D^{(3)}_{n,x}$ are equivalent at the 2^{nd} order.

Remark 7.6 Notice that $\Sigma_2^{D^{(3)}_{n,x}} - \Sigma_2^{C^{(3)}_{n,x}}$ is the D.T. analog of the utility premium in the E.U. model.

7.2 The 4^{rd} Order

Next, we derive a simple class of lottery pairs such that the direction of preference between these lotteries is equivalent to signing the 4^{th} derivative of the probability weighting function. Consider, with $\delta, \delta' > 0$,

$$G^{(4)} = [\ \delta, 1/n\ ;\ \ldots;\ -\delta, 1/n\ ;\ \ldots;\ -\delta, 1/n\ ;\ \ldots;\ \delta, 1/n\ ] \quad \text{and}$$
$$B^{(4)} = [\ -\delta', 1/n\ ;\ \ldots;\ \delta', 1/n\ ;\ \ldots;\ \delta', 1/n\ ;\ \ldots;\ -\delta', 1/n\ ].$$

Upon adding $G^{(4)}$ and $B^{(4)}$ state-wise to an arbitrarily given initial lottery (having at least four states, such that the state-wise addition is feasible), we generate $D^{(4)}$ if $G^{(4)}$ state-wise precedes $B^{(4)}$ and generate $C^{(4)}$ if $B^{(4)}$ state-wise precedes $G^{(4)}$. We note that: (i) δ and δ' need not be equal; (ii) the number of states that "…" denotes may differ within and between $G^{(4)}$ and $B^{(4)}$, but both $G^{(4)}$ and $B^{(4)}$ need to be symmetrical in the sense that the first and third "…" within $G^{(4)}$ and $B^{(4)}$ need to be equal; (iii) the number of states that the first and third "…" denote in $G^{(4)}$ and $B^{(4)}$ is non-negative and may equal zero, while the number of states that the middle (second) "…" denotes is larger than or equal to minus one;[16] (iv) we do not require equal spacing between the two blocks $G^{(4)}$ and $B^{(4)}$ vs. $B^{(4)}$ and $G^{(4)}$ when adding them to the initial lottery and may, in fact, also have negative spacing, so that the two blocks overlap (but with $G^{(4)}$ strictly preceding $B^{(4)}$ for $D^{(4)}$ and $B^{(4)}$ strictly preceding $G^{(4)}$ for $C^{(4)}$). All as long as the initial ranking of outcomes remains unaffected and the outcomes of the resulting lotteries remain non-negative.

Theorem 7.7 *If $C^{(4)}$ and $D^{(4)}$ are according to the above construction, then $D^{(4)}$ is preferred to $C^{(4)}$ by any D.T. D.M. having $h''' \geq 0$, $h'''' \leq 0$.*

Proof. Recall that the probabilities of generating the minimal outcome in a two-shot and three-shot experiment for the states of an n-state lottery are

$$(2n-1)/n^2, \ldots, 7/n^2, 5/n^2, 3/n^2, 1/n^2,$$

and

$$(3(n-1)n+1)/n^3, \ldots, 37/n^3, 19/n^3, 7/n^3, 1/n^3,$$

respectively. Hence, it follows that

$$\mathrm{E}\left[C^{(4)}\right] = \mathrm{E}\left[D^{(4)}\right],$$
$$\mathrm{E}\left[\min(C_1^{(4)}, C_2^{(4)})\right] = \mathrm{E}\left[\min(D_1^{(4)}, D_2^{(4)})\right],$$
$$\mathrm{E}\left[\min(C_1^{(4)}, C_2^{(4)}, C_3^{(4)})\right] = \mathrm{E}\left[\min(D_1^{(4)}, D_2^{(4)}, D_3^{(4)})\right],$$

and $S_{-C^{(4)}}$ surpasses $S_{-D^{(4)}}$ after crossing three times, so that $D^{(4)}$ precedes $C^{(4)}$ in 4^{th} dual stochastic order; see e.g., Proposition 4.9 of Wang and Young [33]. Indeed, adding $G^{(4)}$ and $B^{(4)}$ with $G^{(4)}$ state-wise preceding $B^{(4)}$ and adding $B^{(4)}$ and $G^{(4)}$ with $B^{(4)}$ state-wise preceding $G^{(4)}$ has the same impact on the dual moments up to the 3^{rd} order. □

[16] If the number of states that "…" represents is -1, the middle two adjacent states overlap, adding -2δ for $G^{(4)}$ and $2\delta'$ for $B^{(4)}$ to a state with probability $1/n$.

As in Subsection 7.1, let's consider special subclasses of $C^{(4)}$ and $D^{(4)}$, sufficient for our purpose of signing h''''. Arbitrarily fix (n, x) with $n \geq 4, x \geq 0$ and consider the lottery

$$\begin{aligned} C^{(4)}_{n,x} =&[\ 0, 1/n\ ;\ \ldots;\ x-1, 1/n\ ; \\ &\mathbf{x}, \mathbf{1/n}\ ;\ \mathbf{x+1}, \mathbf{1/n}\ ;\ \mathbf{x+2}, \mathbf{1/n}\ ;\ \mathbf{x+3}, \mathbf{1/n}\ ; \\ &x+4, 1/n;\ \ldots\ ;\ ., 1/n\ ]. \end{aligned}$$

The four states in bold need to be present for any $n(\geq 4)$, the remaining states are added until the state probabilities sum up to 1.

The increments in the outcomes when moving to a higher state for the lottery class $C^{(4)}_{n,x}$ are again normalized to be 1, but may be generalized to be arbitrarily non-negative, as long as the squeezes and anti-squeezes we perform below do not change the initial ranking of outcomes.[17] We notice in particular that increments equal to zero may be permitted, effectively yielding lotteries with unequal state probabilities.

Let $M \geq 4$. Then, adding (state-wise) to the four states in bold

$$C2D^{(4)}_{n,x} = [\ 1/M, 1/n\ ;\ -3/M, 1/n\ ;\ 3/M, 1/n\ ;\ -1/M, 1/n\ ]$$

yields $D^{(4)}_{n,x}$:

$$\begin{aligned} D^{(4)}_{n,x} =&[\ 0, 1/n\ ;\ \ldots;\ x-1, 1/n\ ; \\ &\mathbf{x+1/M}, \mathbf{1/n}\ ;\ \mathbf{x+1-3/M}, \mathbf{1/n}\ ;\ \mathbf{x+2+3/M}, \mathbf{1/n}\ ;\ \mathbf{x+3-1/M}, \mathbf{1/n}\ ; \\ &x+4, 1/n;\ \ldots\ ;\ ., 1/n\ ]. \end{aligned}$$

Lottery $D^{(4)}_{n,x}$ is obtained from lottery $C^{(4)}_{n,x}$ by squeezing the distribution at the lowest two states in bold and equivalently anti-squeezing the distribution at the middle two states in bold; and next equivalently anti-squeezing the distribution at the middle two states in bold and equivalently squeezing the distribution at the highest two states in bold. To keep the required number of states as small as possible, to enhance parsimony of our approach, we again squeeze and anti-squeeze the distribution at overlapping states as in Subsection 7.1, but different from Section 4.

Theorem 7.8 *If $C^{(4)}_{n,x}$ and $D^{(4)}_{n,x}$ are according to the above construction, then, for any $n \geq 4, x \geq 0$, $D^{(4)}_{n,x}$ is preferred to $C^{(4)}_{n,x}$ by any D.T. D.M. having $h''' \geq 0$, $h'''' \leq 0$.*

Proof. The result follows from Theorem 7.7 in view of the fact that $C^{(4)}_{n,x}$ and $D^{(4)}_{n,x}$ are subclasses of $C^{(4)}$ and $D^{(4)}$. □

Theorem 7.9 *If, for any $n \geq 4, x \geq 0$, lottery $D^{(4)}_{n,x}$ is preferred to lottery $C^{(4)}_{n,x}$ by a D.T. D.M. with $h''' \geq 0$, then $h'''' \leq 0$.*

Proof. Since h''' is differentiable on $(0, 1)$, the fourth derivative of h being negative means (is equivalent to requiring) that

$$\begin{aligned} &[([h(p_{++}) - h(p_+)] - [h(p_+) - h(p_0)]) - ([h(p_+) - h(p_0)] - [h(p_0) - h(p_-)])] \\ &\quad - [([h(p_+) - h(p_0)] - [h(p_0) - h(p_-)]) - ([h(p_0) - h(p_-)] - [h(p_-) - h(p_{--})])] \\ &\qquad = h(p_{++}) - 4h(p_+) + 6h(p_0) - 4h(p_-) + h(p_{--}) \leq 0, \end{aligned}$$

[17] This is accomplished by restricting M in the squeezes and anti-squeezes explicated below accordingly.

for any five equidistant points $0 \leq p_{--} \leq p_{-} \leq p_0 \leq p_+ \leq p_{++} \leq 1$. We will show that this is satisfied whenever lottery $D_{n,x}^{(4)}$ is preferred to lottery $C_{n,x}^{(4)}$ for any $n \geq 4, x \geq 0$.

Recall that the D.T. value of a lottery A with n outcomes $x_0 = 0 \leq x_1 \leq \cdots \leq x_n$ is given by

$$\mathrm{V}[A] = \sum_{i=1}^{n} \bar{h}(1 - F_A(x_{i-1}))(x_i - x_{i-1}),$$

with $\bar{h}(p) = 1 - h(1-p)$. Here, $F_A(x_0) = 0$ by convention, so $1 - F_A(x_0) = \bar{h}(1 - F_A(x_0)) = 1$. Hence,

$$\begin{aligned}
\mathrm{V}[C_{n,x}^{(4)}] =& \sum_{i=1}^{j} \bar{h}(1 - F_{C_{n,x}^{(4)}}(x_{i-1}))(x_i - x_{i-1}) + \sum_{i=j+1}^{j+5} \bar{h}(1 - F_{C_{n,x}^{(4)}}(x_{i-1}))(x_i - x_{i-1}) \\
&+ \sum_{i=j+6}^{n} \bar{h}(1 - F_{C_{n,x}^{(4)}}(x_{i-1}))(x_i - x_{i-1}) \\
=:& \Sigma_1^{C_{n,x}^{(4)}} + \Sigma_2^{C_{n,x}^{(4)}} + \Sigma_3^{C_{n,x}^{(4)}},
\end{aligned}$$

with j corresponding to the state preceding the four states in bold that are always present, and similarly for $D_{n,x}^{(4)}$. Notice that

$$\Sigma_1^{C_{n,x}^{(4)}} = \Sigma_1^{D_{n,x}^{(4)}}, \qquad \Sigma_3^{C_{n,x}^{(4)}} = \Sigma_3^{D_{n,x}^{(4)}}.$$

Furthermore,

$$\begin{aligned}
\Sigma_2^{C_{n,x}^{(4)}} =& 1\bar{h}(1 - F(x_j)) + 1\bar{h}(1 - F(x_{j+1})) + 1\bar{h}(1 - F(x_{j+2})) \\
&+ 1\bar{h}(1 - F(x_{j+3})) + 1\bar{h}(1 - F(x_{j+4})),
\end{aligned}$$

and

$$\begin{aligned}
\Sigma_2^{D_{n,x}^{(4)}} =& (1 + 1/M)\bar{h}(1 - F(x_j)) + (1 - 4/M)\bar{h}(1 - F(x_{j+1})) + (1 + 6/M)\bar{h}(1 - F(x_{j+2})) \\
&+ (1 - 4/M)\bar{h}(1 - F(x_{j+3})) + (1 + 1/M)\bar{h}(1 - F(x_{j+4})),
\end{aligned}$$

suppressing the indices $C_{n,x}^{(4)}$ and $D_{n,x}^{(4)}$ for convenience. Define, for given $n \geq 4, x \geq 0$, the probability q, $0 \leq q \leq 1 - 4/n$, such that

$$\begin{aligned}
F(x_j) &= q, & F(x_{j+1}) &= q + 1/n, & F(x_{j+2}) &= q + 2/n, \\
F(x_{j+3}) &= q + 3/n, & F(x_{j+4}) &= q + 4/n, & F(x_{j+5}) &= q + 5/n.
\end{aligned}$$

Then

$$\begin{aligned}
\Sigma_2^{C_{n,x}^{(4)}} =& 1\bar{h}(1 - q) + 1\bar{h}(1 - q - 1/n) + 1\bar{h}(1 - q - 2/n) \\
&+ 1\bar{h}(1 - q - 3/n) + 1\bar{h}(1 - q - 4/n), \\
\Sigma_2^{D_{n,x}^{(4)}} =& (1 + 1/M)\bar{h}(1 - q) + (1 - 4/M)\bar{h}(1 - q - 1/n) + (1 + 6/M)\bar{h}(1 - q - 2/n) \\
&+ (1 - 4/M)\bar{h}(1 - q - 3/n) + (1 + 1/M)\bar{h}(1 - q - 4/n).
\end{aligned}$$

Defining $p_{--}, p_-, p_0, p_+, p_{++}$ such that

$$4/n \leq 1-q = p_{++} \leq 1, \qquad 1-q-1/n = p_+, \qquad 1-q-2/n = p_0,$$
$$1-q-3/n = p_-, \qquad 1-q-4/n = p_{--},$$

it then follows from the arbitrariness of $n \geq 4, x \geq 0$, hence the arbitrariness of $0 \leq q \leq 1-4/n$, that

$$\Sigma_2^{D_{n,x}^{(4)}} - \Sigma_2^{C_{n,x}^{(4)}} = (1/M)\bar{h}(p_{++}) - (4/M)\bar{h}(p_+) + (6/M)\bar{h}(p_0) - (4/M)\bar{h}(p_-) + (1/M)\bar{h}(p_{--}) \geq 0,$$

or equivalently,

$$h(p_{++}) - 4h(p_+) + 6h(p_0) - 4h(p_-) + h(p_{--}) \leq 0,$$

for any five equidistant points $0 \leq p_{--} \leq p_- \leq p_0 \leq p_+ \leq p_{++} \leq 1$, whenever $D_{n,x}^{(4)}$ is preferred to $C_{n,x}^{(4)}$ for any $n \geq 4, x \geq 0$. □

Remark 7.10 If $h'''' = 0$, then $C_{n,x}^{(4)}$ and $D_{n,x}^{(4)}$ are equivalent: $C_{n,x}^{(4)}$ and $D_{n,x}^{(4)}$ are equivalent at the 3$^{\text{rd}}$ order.

Remark 7.11 Notice that $\Sigma_2^{D_{n,x}^{(4)}} - \Sigma_2^{C_{n,x}^{(4)}}$ is the D.T. analog of the utility premium in the E.U. model.

7.3 The m^{th} Order

We now systematically construct simple nested classes of lottery pairs such that the direction of preference between these lotteries is equivalent to signing the m^{th} derivative of the probability weighting function. We start by considering the 1$^{\text{st}}$ dual order and proceed to construct higher dual orders by induction (iteration). For expositional ease, we construct our sequence with the appropriate versions of $C_{n,x}^{(m)}$ and $D_{n,x}^{(m)}$, realizing that they are subclasses of the more general $C^{(m)}$ and $D^{(m)}$, introduced in Subsections 7.1 and 7.2 for $m = 3, 4$.

Arbitrarily fix (n, x) with $n \geq 1$, $x \geq 0$ and consider the lottery

$$C_{n,x}^{(1)} = [\ 0, 1/n\ ;\ \ldots;\ x-1, 1/n\ ;\ \mathbf{x}, \mathbf{1/n}\ ;\ x+1, 1/n;\ \ldots\ ;\ ., 1/n\ ].$$

The state in bold needs to be present for any $n(\geq 1)$, the remaining states are added until the state probabilities sum up to 1. Let $M \geq 1$.

Then, adding (state-wise) to the state in bold

$$C2D_{n,x}^{(1)} = [\ 1/M, 1/n\ ]$$

yields $D_{n,x}^{(1)}$:

$$D_{n,x}^{(1)} = [\ 0, 1/n\ ;\ \ldots;\ x-1, 1/n\ ;\ \mathbf{x + 1/M}, \mathbf{1/n}\ ;\ x+1, 1/n;\ \ldots\ ;\ ., 1/n\ ].$$

Notice that, for any $n \geq 1, x \geq 0$, $D_{n,x}^{(1)}$ is preferred to $C_{n,x}^{(1)}$ by any D.T. D.M. (having $h' \geq 0$).

Taking the m^{th} dual order as a starting point, we construct the class of lottery pairs for the $(m+1)^{\text{th}}$ dual order by induction (iteration) as follows. We assume $n \geq m+1$.

(i) Add to lottery $C_{n,x}^{(m)}$ a state to the states in bold, having an outcome that is 1 unit larger than the largest outcome of the states in bold. This produces the lottery $C_{n,x}^{(m+1)}$.

(ii) Reconsider $C2D_{n,x}^{(m)}$ and use it as a first (preliminary) version of $C2D_{n,x}^{(m+1)}$.

(iii) Add to this first (preliminary) version of $C2D_{n,x}^{(m+1)}$, $C2D_{n,x}^{(m)}$ with its outcomes multiplied by a factor -1, state-wise from the second state onwards. This produces the final version of $C2D_{n,x}^{(m+1)}$.

(iv) Add $C2D_{n,x}^{(m+1)}$ state-wise to the bold states of $C_{n,x}^{(m+1)}$. This produces $D_{n,x}^{(m+1)}$.

We squeeze and anti-squeeze at overlapping states to keep the required number of states minimal.

Let us proceed from the 1st dual order to the 2nd dual order, applying this construction. Arbitrarily fix (n, x) with $n \geq 2$, $x \geq 0$ and consider the lottery

$$\begin{aligned} C_{n,x}^{(2)} =[\ & 0, 1/n\ ;\ \ldots;\ x-1, 1/n\ ; \\ & \mathbf{x, 1/n}\ ;\ \mathbf{x+1, 1/n}\ ; \\ & \qquad x+2, 1/n;\ \ldots\ ;\ ., 1/n\ ]. \end{aligned}$$

The states in bold need to be present for any $n(\geq 2)$, the remaining states are added until the state probabilities sum up to 1. Let $M \geq 2$. Then, adding (state-wise) to the states in bold

$$\begin{aligned} C2D_{n,x}^{(2)} &= [\ 1/M, 1/n\ ;\ 0, 1/n\ ] + [\ 0, 1/n\ ;\ -1/M, 1/n\ ] \\ &= [\ 1/M, 1/n\ ;\ -1/M, 1/n\ ] \end{aligned}$$

yields $D_{n,x}^{(2)}$:

$$\begin{aligned} D_{n,x}^{(2)} =[\ & 0, 1/n\ ;\ \ldots;\ x-1, 1/n\ ; \\ & \mathbf{x+1/M, 1/n}\ ;\ \mathbf{x+1-1/M, 1/n}\ ; \\ & \qquad x+2, 1/n;\ \ldots\ ;\ ., 1/n\ ]. \end{aligned}$$

We next proceed to the 3rd dual order. Arbitrarily fix (n, x) with $n \geq 3$, $x \geq 0$ and consider the lottery

$$\begin{aligned} C_{n,x}^{(3)} =[\ & 0, 1/n\ ;\ \ldots;\ x-1, 1/n\ ; \\ & \mathbf{x, 1/n}\ ;\ \mathbf{x+1, 1/n}\ ;\ \mathbf{x+2, 1/n}\ ; \\ & \qquad x+3, 1/n;\ \ldots\ ;\ ., 1/n\ ]. \end{aligned}$$

The states in bold need to be present for any $n(\geq 3)$, the remaining states are added until the state probabilities sum up to 1. Let $M \geq 3$. Then, adding (state-wise) to the states in bold

$$\begin{aligned} C2D_{n,x}^{(3)} &= [\ 1/M, 1/n\ ;\ -1/M, 1/n\ ;\ 0, 1/n\ ] + [\ 0, 1/n\ ;\ -1/M, 1/n\ ;\ 1/M, 1/n\ ] \\ &= [\ 1/M, 1/n\ ;\ -2/M, 1/n\ ;\ 1/M, 1/n\ ] \end{aligned}$$

yields $D^{(3)}_{n,x}$:

$$\begin{aligned}D^{(3)}_{n,x} =&[\ 0,1/n\ ;\ \ldots;\ x-1,1/n\ ;\\ &\mathbf{x+1/M,1/n}\ ;\ \mathbf{x+1-2/M,1/n}\ ;\ \mathbf{x+2+1/M,1/n}\ ;\\ &x+3,1/n;\ \ldots\ ;\ .,1/n\ ].\end{aligned}$$

We next proceed to the 4[th] dual order. Arbitrarily fix (n,x) with $n \geq 4$, $x \geq 0$ and consider the lottery

$$\begin{aligned}C^{(4)}_{n,x} =&[\ 0,1/n\ ;\ \ldots;\ x-1,1/n\ ;\\ &\mathbf{x,1/n}\ ;\ \mathbf{x+1,1/n}\ ;\ \mathbf{x+2,1/n}\ ;\ \mathbf{x+3,1/n}\ ;\\ &x+4,1/n;\ \ldots\ ;\ .,1/n\ ].\end{aligned}$$

The states in bold need to be present for any $n(\geq 4)$, the remaining states are added until the state probabilities sum up to 1. Let $M \geq 4$. Then, adding (state-wise) to the states in bold

$$\begin{aligned}C2D^{(4)}_{n,x} =&[\ 1/M,1/n\ ;\ -2/M,1/n\ ;\ 1/M,1/n\ ;\ 0,1/n\ ]\\ &+[\ 0,1/n\ ;\ -1/M,1/n\ ;\ 2/M,1/n\ ;\ -1/M,1/n\ ]\\ =&[\ 1/M,1/n\ ;\ -3/M,1/n\ ;\ 3/M,1/n\ ;\ -1/M,1/n\ ]\end{aligned}$$

yields $D^{(4)}_{n,x}$:

$$\begin{aligned}D^{(4)}_{n,x} =&[\ 0,1/n\ ;\ \ldots;\ x-1,1/n\ ;\\ &\mathbf{x+1/M,1/n}\ ;\ \mathbf{x+1-3/M,1/n}\ ;\ \mathbf{x+2+3/M,1/n}\ ;\ \mathbf{x+3-1/M,1/n}\ ;\\ &x+4,1/n;\ \ldots\ ;\ .,1/n\ ].\end{aligned}$$

We next proceed to the 5[th] dual order. Arbitrarily fix (n,x) with $n \geq 5$, $x \geq 0$ and consider the lottery

$$\begin{aligned}C^{(5)}_{n,x} =&[\ 0,1/n\ ;\ \ldots;\ x-1,1/n\ ;\\ &\mathbf{x,1/n}\ ;\ \mathbf{x+1,1/n}\ ;\ \mathbf{x+2,1/n}\ ;\ \mathbf{x+3,1/n}\ ;\ \mathbf{x+4,1/n}\ ;\\ &x+5,1/n;\ \ldots\ ;\ .,1/n\ ].\end{aligned}$$

The states in bold need to be present for any $n(\geq 5)$, the remaining states are added until the state probabilities sum up to 1. Let $M \geq 10$. Then, adding (state-wise) to the states in bold

$$\begin{aligned}C2D^{(5)}_{n,x} =&[\ 1/M,1/n\ ;\ -3/M,1/n\ ;\ 3/M,1/n\ ;\ -1/M,1/n\ ;\ 0,1/n\ ]\\ &+[\ 0,1/n\ ;\ -1/M,1/n\ ;\ 3/M,1/n\ ;\ -3/M,1/n\ ;\ 1/M,1/n\ ]\\ =&[\ 1/M,1/n\ ;\ -4/M,1/n\ ;\ 6/M,1/n\ ;\ -4/M,1/n\ ;\ 1/M,1/n\ ]\end{aligned}$$

yields $D_{n,x}$:

$$\begin{aligned}D^{(5)}_{n,x} =&[\ 0,1/n\ ;\ \ldots;\ x-1,1/n\ ;\\ &\mathbf{x+1/M,1/n}\ ;\ \mathbf{x+1-4/M,1/n}\ ;\ \mathbf{x+2+6/M,1/n}\ ;\ \mathbf{x+3-4/M,1/n}\ ;\ \mathbf{x+4+1/M,1/n}\ ;\\ &x+5,1/n;\ \ldots\ ;\ .,1/n\ ].\end{aligned}$$

We next proceed to the 6$^{\text{th}}$ dual order. Arbitrarily fix (n,x) with $n \geq 6$, $x \geq 0$ and consider the lottery

$$
\begin{aligned}
C^{(6)}_{n,x} =& [\ 0, 1/n\ ;\ \ldots;\ x-1, 1/n\ ;\\
&\mathbf{x, 1/n\ ;\ x+1, 1/n\ ;\ x+2, 1/n\ ;\ x+3, 1/n\ ;\ x+4, 1/n\ ;\ x+5, 1/n}\ ;\\
&x+6, 1/n;\ \ldots\ ;\ ., 1/n\ ].
\end{aligned}
$$

The states in bold need to be present for any $n (\geq 6)$, the remaining states are added until the state probabilities sum up to 1. Let $M \geq 20$. Then, adding (state-wise) to the states in bold

$$
\begin{aligned}
C2D^{(6)}_{n,x} =& [\ 1/M, 1/n\ ;\ -4/M, 1/n\ ;\ 6/M, 1/n\ ;\ -4/M, 1/n\ ;\ 1/M, 1/n\ ;\ 0, 1/n\ ]\\
&+ [\ 0, 1/n\ ;\ -1/M, 1/n\ ;\ 4/M, 1/n\ ;\ -6/M, 1/n\ ;\ 4/M, 1/n\ ;\ -1/M, 1/n\ ]\\
=& [\ 1/M, 1/n\ ;\ -5/M, 1/n\ ;\ 10/M, 1/n\ ;\ -10/M, 1/n\ ;\ 5/M, 1/n\ ;\ -1/M, 1/n\ ]
\end{aligned}
$$

yields $D^{(6)}_{n,x}$:

$$
\begin{aligned}
D^{(6)}_{n,x} =& [\ 0, 1/n\ ;\ \ldots;\ x-1, 1/n\ ;\\
&\mathbf{x+1/M, 1/n\ ;\ x+1-5/M, 1/n\ ;\ x+2+10/M, 1/n}\ ;\\
&\mathbf{x+3-10/M, 1/n\ ;\ x+4+5/M, 1/n\ ;\ x+5-1/M, 1/n}\ ;\\
&x+6, 1/n;\ \ldots\ ;\ ., 1/n\ ].
\end{aligned}
$$

One may continue this construction up to the m^{th} order, with m an arbitrary positive integer.

Theorem 7.12 *If $C^{(m)}_{n,x}$ and $D^{(m)}_{n,x}$ are according to the above construction, then, for any $n \geq m$, $x \geq 0$, $D^{(m)}_{n,x}$ is preferred to $C^{(m)}_{n,x}$ by any D.T. D.M. having $(-1)^{k-1}h^{(k)} \geq 0$, $k = 1, \ldots, m$.*

Proof. One can verify that, by construction, the first $m-1$ dual moments satisfy

$$
\begin{aligned}
\mathrm{E}\left[C^{(m)}_{n,x}\right] &= \mathrm{E}\left[D^{(m)}_{n,x}\right],\\
\mathrm{E}\left[\min(C^{(m)}_{n,x,1}, C^{(m)}_{n,x,2})\right] &= \mathrm{E}\left[\min(D^{(m)}_{n,x,1}, D^{(m)}_{n,x,2})\right],\\
\mathrm{E}\left[\min(C^{(m)}_{n,x,1}, C^{(m)}_{n,x,2}, C^{(m)}_{n,x,3})\right] &= \mathrm{E}\left[\min(D^{(m)}_{n,x,1}, D^{(m)}_{n,x,2}, D^{(m)}_{n,x,3})\right],\\
&\vdots\\
\mathrm{E}\left[\min(C^{(m)}_{n,x,1}, C^{(m)}_{n,x,2}, C^{(m)}_{n,x,3}, \ldots, C^{(m)}_{n,x,m-1})\right] &= \mathrm{E}\left[\min(D^{(m)}_{n,x,1}, D^{(m)}_{n,x,2}, D^{(m)}_{n,x,3}, \ldots, D^{(m)}_{n,x,m-1})\right],
\end{aligned}
$$

and that $S_{-C^{(m)}_{n,x}}$ surpasses $S_{-D^{(m)}_{n,x}}$ after crossing $m-1$ times, so that $D^{(m)}_{n,x}$ precedes $C^{(m)}_{n,x}$ in m^{th} dual stochastic order; see e.g., Proposition 4.9 of Wang and Young [33]. $\square$

Theorem 7.13 *If, for any $n \geq m$, $x \geq 0$, lottery $D^{(m)}_{n,x}$ is preferred to lottery $C^{(m)}_{n,x}$ by a D.T. D.M. with $(-1)^{k-1}h^{(k)} \geq 0$, $k = 1, \ldots, m-1$, then $(-1)^{m-1}h^{(m)} \geq 0$.*

Proof. The proof follows from considering

$$
\Sigma_2^{D^{(m)}_{n,x}} - \Sigma_2^{C^{(m)}_{n,x}},
$$

defined similarly as their analogs in the proofs of Theorems 7.4 and 7.9, and the definition of $(-1)^{m-1}h^{(m)} \geq 0$. $\square$

8 Conclusion

Starting with Menezes, Geiss and Tressler [23], many papers have been devoted to an interpretation of the signs of the successive derivatives of the utility function within the E.U. model. Quite surprisingly there had been so far no analysis of the equivalent problem outside the E.U. framework.

In the present paper we have developed a model free story that is appropriate to satisfy the specific requirements of the D.T. setting.

By doing so, we have also obtained, as a by-product of interest in its own right, a simple and intuitive interpretation of the well-known fact that E.U. and D.T. diverge from the 3$^{\text{rd}}$ order onwards.

We have shown that, while the sign of the 3$^{\text{rd}}$ derivative of the utility function is connected to a savings problem, the sign of the 3$^{\text{rd}}$ derivative of the probability weighting function is naturally linked to a self-protection problem. We have also analyzed implications of our results for portfolio choice, which appear to stand in sharp contrast to familiar implications under the E.U. model.

Because the primal and dual stories have aspects in common, many implications that resulted from the primal story can potentially be extended to a dual world. For instance, because it is also simple, the dual story should be as amenable to experimentation under the D.T. model as the primal story. Another promising feature is that, because D.T. is "orthogonal" to E.U., it can serve as a building block on the basis of which it should be possible to obtain related interpretations for more general models of choice under risk (and ambiguity).

Indeed, now that the dual story has been told, the development of an interpretation to the signs of the successive derivatives of both the utility function and the probability weighting function in the rank-dependent utility (RDU) model of Quiggin [26] is only a small step. It just requires to define similar nested classes of lottery pairs, in such a way that *both* the primal moments *and* the dual moments are affected by the same amount for each lottery pair at each order. This can be achieved by applying the squeezing and anti-squeezing operations defined in this paper to states with outcomes that differ by the same amount for each squeeze and for each anti-squeeze.[18] The resulting pairs of lotteries preserve equality of not only the dual moments (as in this paper) but also the primal moments, and will be such that the direction of preference between them is equivalent to signing the successive derivatives of both the utility function and the probability weighting function of the RDU model. Thus, they can serve to experimentally test hypotheses on higher-order risk attitudes in RDU, as explicitly desired by Deck & Schlesinger [7].

Appendix 1

To recover Mao's lotteries from the dual story instead of from the primal story as in Section 3, one has to start from

$$A = [\ 0, 1/4\ ;\ 2, 1/4\ ;\ 2, 1/4\ ;\ 2, 1/4\ ],$$

and move to

$$\check{B} = [\ 0 + 1/M, 1/4\ ;\ 2 - 1/M, 1/4\ ;\ 2 - 1/M, 1/4\ ;\ 2 + 1/M, 1/4\ ],$$

[18]E.g., at the 3$^{\text{rd}}$ order, in the setting of Section 7.1, this requires that $x_{j+2} - x_{j+1} = x_{j+3} - x_{j+2}$ with j corresponding to the state preceding the three states in bold that are always present.

with $M \geq 1$. By virtue of Theorem 7.2, such a move is approved by all D.T. D.M.'s with $h''' \geq 0$. Now if $M = 1$ holds, $\breve{B}$ becomes B given by

$$B = [\ 1, 1/4\ ;\ 1, 1/4\ ;\ 1, 1/4\ ;\ 3, 1/4\ ],$$

which constitutes, jointly with A above, Mao's lotteries.

To recover the Eeckhoudt and Schlesinger [14] lotteries (3.1) from the dual story, one has to start from

$$\tilde{A} = [\ 2, 1/4\ ;\ 6, 1/4\ ;\ 10, 1/4\ ;\ 10, 1/4\ ],$$

and move to

$$\bar{B} = [\ 2 + 1/M, 1/4\ ;\ 6 - 1/M, 1/4\ ;\ 10 - 1/M, 1/4\ ;\ 10 + 1/M, 1/4\ ],$$

with $M \geq 1/2$. By virtue of Theorem 7.2, such a move is approved by all D.T. D.M.'s with $h''' \geq 0$. Now if $M = 1/2$ holds, $\bar{B}$ becomes $\tilde{B}$ given by

$$\tilde{B} = [\ 4, 1/4\ ;\ 4, 1/4\ ;\ 8, 1/4\ ;\ 12, 1/4\ ],$$

which corresponds to (3.1).

Appendix 2

Consider the lotteries $\breve{A}$ and $\breve{B}$ given by

$$\breve{A} = [\ 2, 1/8\ ;\ 4, 6/8\ ;\ 6, 1/8\ ], \qquad \breve{B} = [\ 3, 1/2\ ;\ 5, 1/2\ ],$$

to be compared at the 4^{th} order. Lotteries $\breve{A}$ and $\breve{B}$ can be generated from the lottery L given by

$$L = [\ 1, 1/2\ ;\ 2, 1/2\ ],$$

upon the apportionment of two "Mao-type" lotteries.

Indeed, start from a lottery $\breve{L}$ given by

$$\breve{L} = [\ 2, 1/2\ ;\ 3, 1/2\ ],$$

and allocate the zero-mean lottery ε given by

$$\varepsilon = [\ -1, 1/2\ ;\ 1, 1/2\ ],$$

to the worst and best outcomes of $\breve{L}$. This generates the Mao-type lotteries $\dot{A}$ and $\dot{B}$ given by

$$\dot{A} = [\ 1, 1/4\ ;\ 3, 3/4\ ], \qquad \dot{B} = [\ 2, 3/4\ ;\ 4, 1/4\ ],$$

respectively. Next, apportioning $\dot{A}$ to the worst outcome of L and $\dot{B}$ to the best outcome of L generates $\breve{A}$, while apportioning $\dot{B}$ to the worst outcome of L and $\dot{A}$ to the best outcome of L generates $\breve{B}$.

Under E.U., $U'''' \leq (\geq)\ 0 \Rightarrow \breve{B} \succeq (\preceq)\ \breve{A}$, in particular $U'''' = 0 \Rightarrow \breve{B} \sim \breve{A}$. What do we know for D.T.? If h is cubic so that $h'''' = 0$, in general $\mathrm{V}[\breve{A}] \neq \mathrm{V}[\breve{B}]$. To verify, let $h(p) = \alpha p - \beta p^2 + \gamma p^3$. Note that $h(0) = 0$. To have $h(1) = 1$, $\alpha - \beta + \gamma = 1$ should hold.

Hence, $h(p) = (1+\beta-\gamma)p - \beta p^2 + \gamma p^3$, so that $h'(p) = 1+\beta-\gamma-2\beta p+3\gamma p^2$. To have $h'(1) \geq 0$ (hence $h'(p) \geq 0$ whenever $h''(p) \leq 0$), $\beta \leq 1+2\gamma$ should hold. Furthermore, $h''(p) = -2\beta + 6\gamma p$. To have $h''(1) \leq 0$ (hence $h''(p) \leq 0$ whenever $h'''(p) \geq 0$), $\gamma \leq (1/3)\beta$ should hold. Finally, $h'''(p) = 6\gamma$, so $h''' \geq 0$ if $\gamma \geq 0$. In sum, $0 \leq \gamma \leq (1/3)\beta$, $(1/2)(\beta-1) \leq \gamma$ and $\alpha = 1+\beta-\gamma$. With such a probability weighting function,

$$\mathrm{V}[\breve{A}] = 4 - (7/16)\beta + (21/32)\gamma, \qquad \mathrm{V}[\breve{B}] = 4 - (1/2)\beta + (3/4)\gamma.$$

Hence, $\mathrm{V}[\breve{A}] - \mathrm{V}[\breve{B}] \geq 0$, with strict inequality whenever $\beta, \gamma > 0$, which means that E.U. and D.T. do not agree at the 4^{th} order in this case.

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