An Equilibrium Model for the OTC Derivatives Market with A Collateral Agreement*

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April 14, 2015

Abstract

In this paper, we consider the over the counter (OTC) derivatives market model with the counterparty risk and the collateral agreement. We then verify the effect of the collateral agreement on the derivative transaction by using the equilibrium analysis in Microeconomics. We first model the financial market as an incomplete market model which forces us to use the utility-based pricing approach. The option and swap contracts are considered in our study. The former and later correspond the unilateral and bilateral counterparty risk case, respectively. We next derive the demand/supply curves for both derivative contracts by agent's utility maximizations. This leads the equilibrium volumes and prices for the derivative contracts, and enables us to observe the influence of the collateral agreement on these. Our numerical results also verify how the market equilibriums for the derivatives change according to the change of the collateral amount through the demand/supply changes.

JEL Classification: G10, G12, G13
Keywords: OTC derivative market, counterparty risk (right-way risk, wrong-way risk), collateral, incomplete market, utility-based pricing, market equilibrium

1 Introduction

In this article, we propose a framework to analyze the derivative contracts in the over the counter (OTC) derivative markets with the counterparty risk and the collateral agreement. The counterparty risk is one of the default risks and the risk that the agents fail to honor payments for the derivative contracts. Brigo and Masetti (2005) and Sorensen and Bollier (1994) assessed the derivative prices incorporated the counterparty risk. After the financial crisis in 2008, the counterparty risks have been concerned from many practitioners and researchers. There are several ways to hedge the counterparty risk, for example, hedging of counterpaty risk with the credit charge (called a credit valuation adjustment: CVA), posting the (cash) collateral and transferring of the OTC transactions to the central clearing counterparty (CCP) (c.f., Gregory (2010)). Many financial institutions have been trying to attack the counterparty risks with these materials. G20 in September 2013 also decided to impose the collateral agreements in the interest rate swap contracts.

*This study was supported by a grant-in-aid from Zengin Foundation for Studies on Economics and Finance.

The collateral agreement is a traditional way to eliminate the risk that the borrower is not able to honor her/his payment in the money markets. The collateral agreement provides a sort of insurance to the counterparty who has positive exposure and will be a cost (i.e., default cost) for the counterparty with negative exposure when she/he defaults for the life of the financial contract. As thought that the collateral agreement avoids the borrower's moral-hazard in the money market, it is natural to think that the collateralization effects on the behaviors of the market participants, and eventually on the financial transactions under the existence of the credit risks (includes the counterparty risk of the derivative contracts). For example, Geanakoplos (1996) showed that the collateral agreement improves Pareto-efficiency in the asset market with the default and increases the asset price. It also demonstrated that the collateral agreement reduces the supply of the claim (Acharya and Bisin, 2011). While their studies consider the abstract asset class and do not limit the asset class of the derivative products, there are various financial products and the counterparty risks are differed by the products. Needles to say, there are two type market participants in the OTC derivative contracts, i.e., the holder of long-position and the agent with short-position. The both counterparties are exposed to the counterparty risk of the swap contract, while the long-holder is only exposed to the counterparty risk of the short-holder for the option contract. That is, it is thought that the impact of the collateralization on the asset market is differed by the financial products. Our research considers the OTC derivative markets and takes into account the difference of the counterparty risks. Furthermore, Acharya and Bisin (2011) analyzed the incentive of the derivative seller to default rather than considered the impact of the collateral on the price and traded amount. That is, they considered the problem of *ex-post* decision making for the agent with the negative exposure. In this study, we remodel the framework of Acharya and Bisin (2011) and derive *ex-ante* market equilibrium. This analysis also makes us available to investigate the effects of the collateral agreement on the price and volume of the claim at the contract date.

There are many derivatives traded in the OTC markets. They include the products whose underlying assets are not traded in the (financial) markets such as the claims written on the stock price indices, weather derivatives, emission trading and so on. To model such derivative markets, the so-called nontraded asset model (or basis-risk model) has been introduced and considered by many researchers (Davis and Yoshikawa 2010, Henderson and Liang 2014, Monoyios 2004, Musiela and Zariphopoulou 2004, Takino 2015, Yamada 2007). We apply the nontraded asset model to describe the OTC market too. The nontraded asset model is one of the incomplete market models in the mathematical finance theory. It is thus required the pricing approaches to obtain unique price of the claim despite of the standard arbitrage pricing method. The derivative pricing models for the nontraded asset models, especially were considered in the context of the utility-based pricing models (Davis and Yoshikawa 2010, Henderson and Liang 2014, Monoyios 2004, Musiela and Zariphopoulou 2004, Yamada 2007) among several methods proposed. These utility-based approaches are based on the certainty equivalent criteria, and provide a so-called reservation price. That is, they give the prices at which the buyer or seller of the claim is willing to trade. On the other hand, the equilibrium pricing models have been proposed up to now as one of pricing approaches with the utility function (Acharya and Bisin 2011, Bessembinder and Lemmon 2002, Biais et al., 2013, Cao and Wei 2004, Dubey et al., 2005, Kijima et al., 2010, Lee and Oren 2009, Takino 2015, W. Lo et al., 2004). *Equilibrium* means the demand-supply balance or *market clearing* in some literatures. Such pricing models provide the prices determined or traded in the market rather than the reservation price. Hence the equilibrium price is able to incorporate the behaviors of both market participants into the price.

The equilibrium pricing models have been mainly concentrated on the derivation of the pricing method for the assets included the derivatives. In the general equilibrium framework, Cao and

Wei (2004) proposed an equilibrium pricing rule represented by the Euler equation. In their work, the price is derived from the utility maximization problem for the representative investor with the market clearing condition of the financial and goods market. Kijima et al., (2010) derived the pricing kernel for the derivative written on the emission. In the partial equilibrium framework, Lee and Oren (2009) derived the equilibrium prices for the weather derivative by the utility maximization problems of the market participants, and demonstrated the hedge performance of the introduction of the derivative contracts. The advantage of the equilibrium approach is not only to give the pricing method leading unique price, but also traded volume for the asset. W. Lo et al., (2004) showed that the increase of the transaction cost decreases the traded volume. In the context of the market model with the counterparty risk, Dubey et al., (2005) showed that the claim volume is reduced due to the counterparty risk. We extend their models to the market model incorporated with the counterparty risk and the collateral agreement. We then observe the influences of the counterparty risk and the collateral agreement on the traded amount of the claim by using the equilibrium pricing approach.

Some researchers have provided the economic analyses of the asset markets including OTC derivative markets with the counterparty risks besides Geanakoplos (1996). At first, the counterparty risk leads to the significant drops in the asset price (c.f., Biais et al., 2013, Krishnamurthy 2010, Takino 2015). Acharya and Bisin (2011) considered the market equilibrium for the contingent claim and verify the role of CCP. Duffie and Zhu (2011) have shown that the CCP eases the counterparty risk exposure for a particular derivative category. As to the credit charge (CVA), Takino (2015) considered the OTC option market with counterparty risk and shows the impact of the credit charge on the market equilibrium for the option. He showed that the credit charge has an effect to maintain the liquidity of the derivative. However, to our best knowledge, there are few previous studies about the effects of the collateralization on the OTC derivative markets. Our study shows that in the market equilibrium point of view and verifies the effect of the collateralization on the market.

We examine the sensitivity analyses for the market equilibriums with respect to the collateral amount to investigate the influences of the collateralization on the derivatives trading. That is, we observe how the equilibrium price and traded amount (or volume) change when the collateral amount changes. We further consider the option contract and swap contract. The former corresponds to the case that only one counterparty has the counterparty risk (we call unilateral risk) and the latter is related to the case that both counterparties have the counterparty risk each other (we call bilateral risk). Therefore, our study does not only verify the effect of the collateralization on the market equilibrium, and also the differences of impacts by the unilateral/bilateral counterparty risks. At this point, the relation between the collateralization and the derivative price has been considered in some literatures (Fujii and Takahashi 2013, Johannes and Sundaresan 2003). The risk-neutral pricing methods were used in their studies. We also demonstrate whether the relation between the collateralization and the derivative price in our equilibrium pricing approach consists with the one in the risk-neutral pricing rule. We show that: The equilibrium price and volume are affected by the collateral amount. The relation between the derivative price and the collateral amount for the option coincides with the one via the risk-neutral pricing. As to the impact on the traded volume, we observe that the collateralization tends to reduce the traded volume, whereas it has a possibility to recover the liquidity for a large collateral amount even if either or both market participants have the wrong-way risk for the counterparty's default. Furthermore, the influence of the collateralization on the derivative trade in the unilateral counterparty risk is more significant than the one in the bilateral risk. That is, for the derivative contract with bilaterally arising the counterparty risk like a swap, the collateralization reduces the counterparty risk maintaining the market level measured by the traded price and volume. This result supports the decision of G20 in September 2013.

The remainder of the paper is organized as follows: In Section 2, we set the financial market model and the behaviors of the market participants and the collateral agreement. And also, we provide derivative prices from the risk-neutral pricing method to compare our equilibrium pricing formula. In Section 3, we derive the equilibrium price and volume for the option contract as an example of the unilateral counterparty risk case. The sensitivity analyses of the price and volume with respect to the collateral amount are also provided. In Section 4, we derive the equilibrium swap rate and volume of the swap contract as an example of the bilateral counterparty risk case. In Section 5, we observe the effects of the collateralization on the price and volume for both derivative contracts by the numerical way. The work is concluded in Section 6.

2 Model and Collateralization

In this section, we introduce the framework to analyze the impact of the counterparty risk and the collateral agreement on the OTC derivatives market. We use the demand-supply analysis for the claim. We first set the market model includes the counterparty risk and then derive the demand/supply functions via the (static) utility maximization problem.

2.1 Financial Markets

There are two market participants in our financial market, they respectively invest their money to the risky businesses. We denote by S_t^j the risky business value at time t $(0 \leq t \leq T)$ invested from the participant j where T denotes the maturity date of the derivatives introduced in the following. We assume that the risky business is traded in the large market and the agents can invest her/his money into the risky business at the unit price S_t^j at time t. Note that we regard the risky businesses as a sort of the traded asset. The values of the risky businesses are correlated with a common asset price which is assumed to be *nontraded* in the market and the price process is denoted by $\{Y_t\}_{0\leq t\leq T}$. For instance, the price indices itself of the stock markets are not traded in the financial market though it is natural to think that many company's earnings are correlated with it. As to another example, the weather or energy indices are related to our market model (e.g., Bessembinder and Lemmon (2002), Cao and Wei (2004), Kijima et al., (2010), Lee and Oren (2009), Yamada (2007)). We especially do not specify what Y means.

The market participants are also supposed to trade the European type derivatives written on Y to hedge the risk of the business earnings and its payoff function at the maturity T is defined as

$$H(T) := H(T, Y_T).$$

We consider the call option contract and the swap contract, and assume that those contracts are entered at time 0. The assumption of "nontraded" of Y leads that our derivative market is incomplete. That is, the value of $H(T)$ is not perfectly hedged by traded assets and does not enables us to use the risk-neutral pricing such as Black-Scholes pricing formula to determine unique derivative price. The market participants are divided two types, that is, the one takes the long positions (denoted by "l") and the other has the short positions (denoted by "s") in the derivative contracts. We suppose that they behave as the *price taker* in the financial markets included the derivative market. The remaining money is deposited into the bank account with the interest rate r (i.e., risk-free asset). The value of the bank account at time t is $B_t = e^{rt}$ with $B_0 = 1$.

The counterparty risk in the derivative contract is the possibilities that the participants fail to full-payout of $H(T)$. We assume that the default payment depends on the counterpaty's business value at the maturity as introduced by Henderson and Liang (2014). The payment which the receiver $j \in \{l, s\}$ can obtain, is then represented by $\eta_i(S_T^i)H(T)$ $(i \neq j)$ when the counterparty i defaults where $\eta_i(\cdot)$ is the recovery function for the participant i's default. Now, for the option contract, we suppose that the buyer of the option does not fail to pay the option fee when its contract is entered. And also, there are possibilities for both counterparties to fail to pay for the swap contract. The case of the option contract and the swap contract in our study, thus correspond to the case of the unilateral couterparty risk and the bilateral counterparty risk, respectively.

There are two ways of modeling the default events, i.e., the reduced form and the constructed form model. The modeling the default events identifies the form of the default indicator function 1_{D_j} $(j = l, s)$. We use the constructed form model for the both option and swap contract as used Henderson and Liang (2014). The constructed model which initiated by Merton (1974) and has been applied in the credit risk modeling, the default of the firm is caused when its asset amounts below a certain level (usually used the liability amounts). The counterparty risk will be precisely modeled in the following sections.

2.2 Collateral Agreement

To hedge the loss due to the counterparty risk, the agent who has a positive exposure could receive the cash collateral from the counterparty who has a negative exposure. The posted collateral amount is determined by the marked-to-market (MtM) of the derivative contract, according to the coverage ratio ϕ (≥ 0). We set the date of MtM by t $(0 \leq t < T)$ and denote the value of the MtM at time t by V_t. Then the value of the collateral amount is given by

$$C(\phi) = \phi V_t. \tag{2.1}$$

We suppose that V_t $(0 \leq t < T)$ is calculated by the naive risk-neutral pricing for convenience. We determine the risk-neutral measure in an ad-hoc way, although it is difficult for us to have a unique pricing measure since our market model is incomplete.

The introduction of the coverage ratio means that the counterparty j who has right to receive $H(T)$ gives up to cover the loss of $(1 - \eta_i(S_T^i))H(T) - C(\phi)$ when the counterparty i is default. Of cause, there is a possibility to result in $C(\phi) > (1 - \eta_i(S_T^i))H(T)$. We suppose that the collateral receiver is allowed not to return the residual cash amounts $C(\phi) - (1 - \eta_i(S_T^i))H(T)$ in that case.

We also suppose that the collateral rate is applied in the collateral agreement. The receiver of the collateral has to return the posted one to the counterparty with interest when she/he will not have defaulted by the maturity. This interest rate is called the collateral rate, we denote by r_c it. We assume that the collateral rate is constant for time horizon $[0, T]$. As a more precise explanation about the coverage ratio and the collateral rate, we refer the readers to Fujii and Takahashi (2013). Finally, we suppose that the both counterparties do not fail to return posted collateral at all.

2.2.1 Option Payoff with Collateral Agreement

The buyer (or long-holder) of the option always has positive exposure. So she/he requires the seller to post the collateral C per unit of claim when the derivative contract is entered. We assume that the collateral for the option contract is delivered on the contract date $(t = 0)$ only, after that, no cash collateral is posted/received up to the maturity, i.e., the collateral amount is

$$C(\phi) = \phi V_0.$$

Under these settings, we provide the payoff $g_{opt}(T)$ of the option subject to the collateral agreement at the maturity. We formulate the value of $g(T)$ from the point of the long-holder, i.e.,

$$g_{opt}(T) = (H(T) - B^c_T C(\phi))(1 - 1_{D_s}) + \eta_s(S^s_T)H(T)1_{D_s} \tag{2.2}$$

where $B^c_T = e^{r_c T}$ and 1_{D_j} $(j = l, s)$ denotes the default indicator function for the counterparty j such that $1_{D_j} = 1$ if the participant j is in the default and $1_{D_j} = 0$ otherwise. The buyer of the option is able to obtain $H(T)$ and simultaneously should return posted collateral with the collateral rate if the seller will not default up to the maturity, the first term shows this fact. The second term corresponds to the default payment of the option. As to the short holder, the formula is given by adding minus sign to g_{opt}.

2.2.2 Swap Payoff with Collateral Agreement

The counterparty risk arises from both sides in the swap contract opposite to the option contract. The standard swap valuation determines the swap price such that the present value of the contract equals to zero. This implies that the exposures of the derivative contract for both couterparties are vanished. As introduced in Johannes and Sundaresan (2003), we consider the two period model. That is, the claim contract is entered at time 0 with the maturity T and MtM is held at the fixed time between $(0, T)$. We set the date of MtM by t, and suppose that MtM is done once. Then, the collateral amount at the MtM date t is given by

$$C(\phi) = \phi V_t.$$

At the date of MtM, the participant who has a positive exposure receives the cash collateral and the agent with negative exposure has to post the collateral. Therefore, the payoff $g_{swp}(T)$ of this swap contract to the long-holder and the collateral agreement is represented by

$$\begin{aligned} g_{swp}(T) = &\left(Y_T - \frac{B^c_T}{B^c_t} C(\phi) 1_{V_t \geq 0} \right)(1 - 1_{D_s}) + \eta_s(S^s_T) Y_T 1_{D_s} \\ &- \left(K + \frac{B^c_T}{B^c_t} C(\phi) 1_{V_t < 0} \right)(1 - 1_{D_l}) - \eta_l(S^l_T) K 1_{D_l}. \end{aligned} \tag{2.3}$$

where t is the date of MtM. We rewrite (2.3) as

$$g_{swp}(T) = g_Y(T) - K g_\perp(T)$$

where

$$\begin{aligned} g_Y(T) &= Y_T \left\{ 1 - (1 - \eta_s(S^s_T))\, 1_{D_s} \right\} - \frac{B^c_T}{B^c_t} \phi V_t \{ 1_{V_t \geq 0}(1 - 1_{D_s}) + 1_{V_t < 0}(1 - 1_{D_l}) \}, \\ g_\perp(T) &= 1 - \left(1 - \eta_l(S^l_T)\right) 1_{D_l}. \end{aligned} \tag{2.4}$$

The long-holder receives Y_T and return posted collateral if she/he has the positive exposure at date of MtM and the short-holder does not default. The first term reflects this. The second term is the default payment from the short-holder. The third term means that the long-holder has to pay K and recover posted collateral if she/he has the negative exposure at MtM date and will have never defaulted. The last term is the default payment from the long-holder. As to the short holder, the formula is given by adding minus sign to g_{swp}.

2.3 Benchmark Pricing

In this section, we provide the standard pricing formulae for both derivative contracts as benchmark formulae to compare our pricing formulae. The risk-neutral pricing formula is provided as the standard formula. We also examine the sensitivity analyses of the price with respect to the coverage ratio. To this end, we suppose in this section that there exists a unique risk-neutral martingale measure $\tilde{\mathbb{P}}$, and we denote the expectation operator under this measure by $\tilde{E}$.

2.3.1 Option Price

From (2.2), the risk-neutral price of the option with the collateral agreement is

$$\tilde{E}\left[\frac{1}{B_T}g_{opt}(T)\right] = \frac{1}{B_T}\tilde{E}[H(T)\{1-(1-\eta_s)1_{D_s}\}] + \frac{B_T^c}{B_T}\phi V_0 \tilde{E}[1_{D_s}].$$

Differentiating $\tilde{E}[g_{opt}(T)]$ with respect to ϕ, we have

$$\frac{\partial \tilde{E}[B_T^{-1}g_{opt}(T)]}{\partial \phi} = \frac{B_T^c}{B_T}V_0\tilde{E}[1_{D_s}] \geq 0$$

from $V_0 \geq 0$ for the option contract. Hence the option price monotonically increases with the increase of the coverage ratio (or collateral amount).

2.3.2 Swap Rate

The risk-neutral swap rate with the collateral agreement solves to

$$\tilde{E}[g_{swp}(T)] = 0$$

with the assumption of constant interest rate. From (2.3), the risk-neutral swap rate is given by

$$K = \frac{\tilde{E}[Y_T\{1-(1-\eta_s)1_{D_s}\}]}{\tilde{E}[1-(1-\eta_l)1_{D_l}]} - \frac{B_T^c}{B_t^c}\frac{\phi\tilde{E}[V_t(1_{V_t\geq 0}(1-1_{D_s}) + 1_{V_t<0}(1-1_{D_l})]}{\tilde{E}[1-(1-\eta_l)1_{D_l}]}.$$

Differentiating K with respect to ϕ, we have

$$\frac{\partial K}{\partial \phi} = -\frac{B_T^c}{B_t^c}\frac{\tilde{E}[V_t(1_{V_t\geq 0}(1-1_{D_s}) + 1_{V_t<0}(1-1_{D_l}))]}{\tilde{E}[1-(1-\eta_l)1_{D_l}]}.$$

Since $\tilde{E}[1-(1-\eta_l)1_{D_l}] \geq 0$, the swap rate monotonically increases with the increase of the coverage ratio if $\tilde{E}[V_t 1_{V_t\geq 0}(1-1_{D_s})] < -\tilde{E}[V_t 1_{V_t<0}(1-1_{D_l})]$, and the swap rate monotonically decreases with the increase of the coverage ratio otherwise. The amount of $\tilde{E}[V_t 1_{V_t\geq 0}(1-1_{D_s})]$ increases when the positive exposure of the long-holder increases and the possibility of the short-holder's default decreases. That is, this amount indicates a sort of the right-way risk which the long-holder receives. On the other hand, the amount of $-\tilde{E}[V_t 1_{V_t<0}(1-1_{D_l})]$ implies a sort of the right-way risk which the short-holder has. Therefore, the relation between the swap rate and the collateral amount depends on the degree of the right-way risk for both sides. We will check whether this characteristic holds for our equilibrium approach in the below.

2.4 Participant's Total Wealth

We provide the total wealth for both participants at each time under the above settings. The agent j has the initial wealth x_0^j and allocates it to the risky business and the derivative contract. The rest of money is deposited to the bank account with a constant interest rate r. We note that either party posts/receives the collateral according to MtM. The participant with the negative exposure pays the cash collateral from her/his wealth and the agent received the collateral deposits it the bank account. The money amount invested in the risky businesses is denoted by $\pi_j := \pi_j(0)$ $(j = l, s)$, this is supposed to be optimized without any derivative contracts. The assumption of the optimization of π reflects that the participants originally invested their money into the risky business in an optimal, after that they decide to enter the derivative contracts. And also, the optimization of π can eliminate the dependence of the parameter S_0^j on the equilibriums in the numerical implementation. We further assume that the agents do not change the position π_j at all for $(0, T]$. The volume or the position of the claim which the participant j is willing to trade, is denoted by k_j.

2.4.1 Total Wealth for Option Contract

We first give the wealth equations when the participants trade the option. We denote by $X^j(t)$ the total wealth at time t for the agent $j = l, s$. The market participant j trades k_j-units of claim at price p per unit of claim. Recall that the counterparty who has a positive exposure is the buyer. The seller must post the collateral to the buyer when the derivative contract is entered. The money amount w_0^j held in the bank account at $t = 0$ is given by

$$w_0^j = x_0^j - \frac{\pi_j}{S_0^j} S_0^j - \delta_j k_j (p - C(\phi)) = x_0^j - \pi_j - \delta_j k_j (p - C(\phi))$$

where $\delta_j = 1$ for the case of the participant $j = l$, and $\delta_j = -1$ otherwise. Then, the total wealth for the participant j at the maturity is represented by

$$X^j(T) = w_0^j B_T + \frac{\pi_j}{S_0^j} S_T^j + \delta_j k_j g_{opt}(T) = (x_0^j - \pi_j - \delta_j k_j (p - C(\phi))) B_T + \frac{\pi_j}{S_0^j} S_T^j + \delta_j k_j g_{opt}(T) \quad (2.5)$$

where $g_{opt}(T)$ is given in (2.2).

2.4.2 Total Wealth for Swap Contract

We next provide the wealth equation subject to the swap contract. The wealth of the participants is consisted with the risk-free asset, the risky business and the derivative position with the collateral agreement as the option case. The differences from the option case are the value of the derivative contract at the initial and the treatment of the collateral. In the swap contract, the exposure is zero at the contract date and the collateral is settled at the interval time between $(0, T)$ as introduced in the above.

Since the value of the swap is zero at the contract time, the money amount w_0^j held in the bank account is

$$w_0^j = x_0^j - \frac{\pi_j}{S_0^j} S_0^j = x_0^j - \pi_j.$$

Recall that, MtM is assumed to be done at time $t \in (0, T)$, the long-holder (short-holder) receives (posts) the cash collateral if the value of MtM is positive (negative) for the long-holder (short-holder) and posts (receives) the collateral otherwise. The received cash amount as collateral is deposited into the bank account. Thus, the money amount w_t^j at time t held in the bank account is

$$w_t^j = B_t w_0^j + \delta_j k_j C(\phi).$$

Then, the total wealth for the participant j at the maturity T is given by

$$\begin{aligned} X^j(T) =& \frac{B_T}{B_t} w_t^j + \frac{\pi_j}{S_0^j} S_T^j + \delta_j k_j g_{swp}(T) \\ =& B_T(x_0^j - \pi_j) + \delta_j k_j \frac{B_T}{B_t} C(\phi) + \frac{\pi_j}{S_0^j} S_T^j + \delta_j k_j g_{swp}(T) \\ =& B_T(x_0^j - \pi_j) + \delta_j k_j \left(\frac{B_T}{B_t} \phi V_t + g_{swp}(T) \right) + \frac{\pi_j}{S_0^j} S_T^j \end{aligned} \tag{2.6}$$

where $g_{swp}(T)$ is given in (2.3).

2.5 Behaviors of the Market Participants

The preferences of the market participants for the risks are assumed to be represented by the expected utility function. We assume that their utilities are *mean-variance* utility, i.e., for the wealth X, the expected utility is formulated as

$$E[U^j(X))] = E[X] - \frac{\gamma_j}{2} Var[X], \quad j = l, s \tag{2.7}$$

where γ_j is the risk-aversion parameter for the participant j.

As introduced in the above, there are two optimizations problems for agents. The first is the problem to decide the investment policy to the risky business without any claims, which is for the technical purpose rather than essential in our model, that is, we consider the problem to maximize the expected utility for the business portfolio consists with the bank account and the risky business. Therefore, we solve the problem

$$E\left[U^j \left((x_0^j - \pi_j) B_T + \frac{\pi_j}{S_0^j} S_T^j \right) \right] \longrightarrow \text{maximize w.r.t. } \pi_j. \tag{2.8}$$

From the first order condition with (2.7), we have the solution of (2.8) as

$$\pi_j^* = -\frac{(S_0^j)^2 \left(B_T - \frac{1}{S_0^j} E[S_T^j] \right)}{\gamma_j Var[S_T^j]}. \tag{2.9}$$

The second is problems to derive the demand/supply for the claims with π^*, that is, we optimize the claim volume to maximize the expected utility from the participant's wealth $X^j(T)$ which given in the previous section. The objective for the participant is then given by

$$E[U^j(X^j(T))] \longrightarrow \text{maximize w.r.t. } k_j.$$

The expected wealth and the variance of it are

$$E[X^j(T)] = (x_0^j - \pi_j - \delta_j k_j (p - \phi V_0)) B_T + \frac{\pi_j}{S_0^j} E[S_T^j] + \delta_j k_j E[g_{opt}(T)], \tag{2.10}$$

$$Var[X^j(T)] = \left(\frac{\pi_j}{S_0^j}\right)^2 Var[S_T^j] + k_j^2 Var[g_{opt}(T)] + 2\delta_j \frac{\pi_j}{S_0^j} k_j Cov[S_T^j, g_{opt}(T)] \tag{2.11}$$

for the option contract, and then for the swap contract

$$E[X^j(T)] = (x_0^j - \pi_j)B_T + \delta_j k_j \phi E[V_t]\frac{B_T}{B_t} + \frac{\pi_j}{S_0^j}E[S_T^j] + \delta_j k_j E[g_{swp}(T)], \tag{2.12}$$

$$Var[X^j(T)] = k_j^2 Var\left[\frac{B_T}{B_t}\phi V_t + g_{swp}(T)\right] + \left(\frac{\pi_j}{S_0^j}\right)^2 Var[S_T^j] + 2\delta_j k_j \frac{\pi_j}{S_0^j} Cov\left[\frac{B_T}{B_t}\phi V_t + g_{swp}(T), S_T^j\right]. \tag{2.13}$$

The optimal k_j's will be solved in the following sections.

3 Unilateral Counterparty Risk Model

In this section, we derive the equilibrium price and volume for the option with collateral agreement as an example of the unilateral counterparty risk case.

3.1 Demand/Supply Function and Market Equilibrium

3.1.1 Demand/Supply Function

The market participant j $(j = l, s)$ determine the claim volume k to maximize the her/his utility, i.e., the optimization problem is formulated as

$$\max_{k_j} E[U^j(X^j(T))].$$

From (2.7) with (2.10) and (2.11), the first order condition is given by

$$-\delta_j(p - \phi V_0))B_T + \delta_j E[g_{opt}(T)] - \gamma_j\left(k_j Var[g_{opt}(T)] + \delta_j \frac{\pi_j}{S_0^j} Cov[S_T^j, g_{opt}(T)]\right) = 0. \tag{3.1}$$

From (3.1), we obtain the optimal k_j^*,

$$\begin{aligned} k_j^* := k_j^*(p;\phi) &= -\frac{\delta_j B_T}{\gamma_j Var[g_{opt}(T)]}p + \delta_j \frac{E[g_{opt}(T)] - \gamma_j/S_0^j \pi_j Cov[S_T^j, g_{opt}(T)]}{\gamma_j Var[g_{opt}(T)]} + \frac{\delta_j B_T}{\gamma_j Var[g_{opt}(T)]}\phi V_0 \\ &=: -\alpha_j p + \beta_j + \alpha_j \phi V_0 \end{aligned} \tag{3.2}$$

for $j = l, s$, where

$$\alpha_j = \frac{\delta_j B_T}{\gamma_j Var[g_{opt}(T)]}, \quad \beta_j = \delta_j \frac{E[g_{opt}(T)] - \gamma_j \pi_j / S_0^j Cov[S_T^j, g_{opt}(T)]}{\gamma_j Var[g_{opt}(T)]}.$$

From (3.2), we have

$$p_j := p_j(k;\phi) = -\frac{1}{\alpha_j}k + \frac{\beta_j}{\alpha_j} + \phi V_0$$

for any k.

This equation gives the demand/supply functions, that is, the demand function for $j = l$ and the supply function for $j = s$. Since α_l is positive, the demand function takes a shape of the right-decreasing on a k-p plane and the supply function has the right-increasing shape since α_s is negative. Also, the money amounts g_{opt} the buyer (seller) receives (pays) depends on the coverage ratio ϕ. Hence, the collateral agreement does not only makes the demand and supply functions shift to upward and downward respectively, but also changes the gradients of the demand/supply curves.

3.1.2 Market Equilibrium

Let us define the equilibrium. The equilibrium in our study means so-called demand-supply equilibrium, that is, the price is determined to balance the demand and supply in the market. That is, the equilibrium price p^* solves to

$$k_l(p;\phi) = k_s(p;\phi).$$

Substituting (3.2) into the equilibrium condition, we have the equilibrium price

$$\begin{aligned} p^* =& \frac{\beta_l - \beta_s}{\alpha_l - \alpha_s} + C(\phi) \\ =& E[B_T^{-1} g_{opt}(T)] - \frac{\gamma_l \gamma_s B_T^{-1}}{\gamma_l + \gamma_s} \left(\frac{\pi_l}{S_0^l} Cov[S_T^l, g_{opt}(T)] + \frac{\pi_s}{S_0^s} Cov[S_T^s, g_{opt}(T)] \right) + \phi V_0. \end{aligned} \tag{3.3}$$

As to the equilibrium volume k^*, it is obtained by plugging (3.3) into (3.2), i.e.,

$$k^* = \frac{\alpha_l \beta_s - \alpha_s \beta_l}{\alpha_l - \alpha_s} = \frac{\gamma_s \pi_s / S_0^s Cov[S_T^s, g_{opt}(T)] - \gamma_l \pi_l / S_0^l Cov[S_T^l, g_{opt}(T)]}{(\gamma_l + \gamma_s) Var[g_{opt}(T)]}. \tag{3.4}$$

3.1.3 Sensitivity Analysis

We observe the impacts of the collateral agreement on the equilibrium price and volume, respectively.

Impact on Equilibrium Price

The partial derivative of p^* in (3.3) with respect to ϕ is calculated as

$$\frac{\partial p^*}{\partial \phi} = \frac{B_T^c}{B_T} V_0 \left\{ E[1_{D_s}] - \frac{\gamma_l \gamma_s}{\gamma_l + \gamma_s} Cov[S_T^s, 1_{D_s}] + \frac{B_T}{B_T^c} - 1 \right\}. \tag{3.5}$$

Unfortunately, we cannot determine the sign of this partial derivative with no condition. The first and second terms in (3.5) are all positive, and the remaining term $B_T/B_T^c - 1$ is nonnegative if $r \geq r_c$ or nonpositive otherwise. That is, (3.5) implies that the equilibrium price has possibility to decrease with the increase of the collateral according to the levels of r and r_c (in practice $B_T/B_T^c - 1$ is negative if $r_c > r$). We are going to precisely check the sign of (3.5) in the numerical example.

Impact on Equilibrium Volume

Next we consider the impact of the collateral on the liquidity. For convenience, we rewrite the equilibrium volume formula as follows,

$$k^* = \frac{\gamma_s \pi_s / S_0^s Cov[S_T^s, g_{opt}(T)] - \gamma_l \pi_l / S_0^l Cov[S_T^l, g_{opt}(T)]}{(\gamma_l + \gamma_s) Var[g_{opt}(T)]} =: \frac{J(\phi)}{I(\phi)}.$$

Lemma 3.1. *The following assertions hold.*

1. *$I(\phi)$ is a convex quadratic function of ϕ.*
2. *$J(\phi)$ is the decreasing linear function of ϕ.*

In fact, the variance of $g_{opt}(T)$ is calculated as

$$\begin{aligned} Var[g_{opt}(T)] =& Var[H(T)(1 - 1_{D_s}) + \eta(S_T^s)H(T)1_{D_s}] + (B_T^c \phi V_0)^2 Var[1 - 1_{D_s}] \\ & - 2B_T^c \phi V_0 Cov[H(T)(1 - 1_{D_s}) + \eta(S_T^s)H(T)1_{D_s}, 1 - 1_{D_s}] \\ =& Var[H(T)(1 - 1_{D_s}) + \eta(S_T^s)H(T)1_{D_s}] + (B_T^c \phi V_0)^2 Var[1_{D_s}] \\ & - 2B_T^c \phi V_0 Cov[H(T)(1 - 1_{D_s}), 1 - 1_{D_s}]. \end{aligned}$$

On the other hand, the covariance between S_t^j and $g_{opt}(T)$ is given as

$$\begin{aligned} Cov[S_t^j, g_{opt}(T)] =& E[S_T^j, g_{opt}(T)] - E[S_T^j]E[g_{opt}(T)] \\ =& Cov[S_T^j, H(T)(1 - 1_{D_s}) + \eta(S_T^s)H(T)1_{D_s}] - Cov[S_T^j, B_T^c \phi V_0 (1 - 1_{D_s})] \\ =& Cov[S_T^j, H(T)(1 - 1_{D_s}) + \eta(S_T^s)H(T)1_{D_s}] + B_T^c \phi V_0 Cov[S_T^j, 1_{D_s}]. \end{aligned}$$

From the assumption, $Cov[S_T^l, 1_{D_s}] = 0$. Hence we have

$$\begin{aligned} & Cov[S_t^l, g_{opt}(T)] = Cov[S_T^l, H(T)(1 - 1_{D_s}) + \eta(S_T^s)H(T)1_{D_s}], \\ & Cov[S_t^s, g_{opt}(T)] = Cov[S_T^s, H(T)(1 - 1_{D_s}) + \eta(S_T^s)H(T)1_{D_s}] + B_T^c \phi V_0 Cov[S_T^s, 1_{D_s}]. \end{aligned}$$

The fact that $Cov[S_T^s, 1_{D_s}] < 0$ completes second assertion of Lemma 3.1.

Lemma 3.1 indicates that, under the *suitable condition*, there exists the optimal $\tilde{\phi}$ which maximizes the equilibrium volume k^*, and the volume increases with the increase of ϕ for $\phi \leq \tilde{\phi}$ and decreases after that. The suitable condition is related to the parameters. More formally, the equilibrium volume k^* increases when $I'(\phi) < J'(\phi)$ and decreases if $I'(\phi) > J'(\phi)$ with increasing of ϕ. The optimal coverage ratio $\tilde{\phi}$ therefore is achieved at

$$I'(\phi) = J'(\phi). \tag{3.6}$$

The form of the equilibrium volume as a function of the coverage ratio considered in the above arguments, is interpreted as follows. The relation especially has two aspects. The one is that the collateral agreement gives a sense of safety to the buyer that she/he can obtain a certain payment if the seller defaults. This increases the demand for the claim and leads growth in the traded amount. The other is that the collateral agreement spoils the moral-hazard of the seller due to the price. It is well known in the standard financial theory that the collateral has a role to avoid the moral hazard of the borrower of the money As demonstrated in Acharya and Bisin (2011), the collateral agreement reduces the supply of the claim compared with the no default case in the derivative market. By contrast to their statement, the moral hazard is avoided by contradiction of the demand for the claim in our study. The collateral agreement raises the claim price which decreases demand for it. In addition, the reduction of the traded volume is interpreted in the point of the transaction cost view. W. Lo et al., (2004) showed that the increment of the transaction cost reduces the traded amount of the asset by a dynamic equilibrium analysis. Applying their result to our result, the collateral has possible to be a "transaction cost" too. In fact, the collateral rises the equilibrium price according to the coverage ratio.

4 Bilateral Counterparty Risk Model

In this section, we derive the equilibrium price (or swap rate) and volume for the swap with collateral agreement as an example of the bilateral counterparty risk case.

4.1 Demand/Supply Function and Market Equilibrium

4.1.1 Demand/Supply Function

The agent's problem is to maximize her/his expected utility for the wealth including the derivative contract, i.e.,

$$\max_{k_j} E[U^j(X^j(T))].$$

From (2.7) with (2.12), (2.13) and the first order condition, we have

$$\begin{aligned}\delta_j \frac{B_T}{B_t}\phi E[V_t] + \delta_j E[g_{swp}(T)] \\ - \gamma_j \left(k_j Var\left[\frac{B_T}{B_t}\phi V_t + g_{swp}(T)\right] + \delta_j \frac{\pi_j}{S_0^j} Cov\left[\frac{B_T}{B_t}\phi V_t + g_{swp}(T), S_T^j\right]\right) = 0. \quad (4.1)\end{aligned}$$

From (4.1), we obtain the optimal k_j^*,

$$k_j^* := k_j^*(K;\phi) = \delta_j \frac{E[g_{swp}(T)] - \gamma_j \frac{\pi_j}{S_0^j} Cov[\frac{B_T}{B_t}\phi V_t + g_{swp}(T), S_T^j] + \frac{B_T}{B_t}\phi E[V_t]}{\gamma_j Var[\frac{B_T}{B_t}\phi V_t + g_{swp}(T)]}. \quad (4.2)$$

(4.2) is rewritten as

$$\begin{aligned}k_j^* = \frac{\delta_j}{\gamma_j Var[\frac{B_T}{B_t}\phi V_t + g_{swp}(T)]} \left\{ E[g_Y(T)] - \gamma_j \frac{\pi_j}{S_0^j} \left(\frac{B_T}{B_t}\phi Cov[S_T^j, V_t] + Cov[S_T^j, g_Y(T)] \right) + \frac{B_T}{B_t}\phi E[V_t] \right\} \\ - \frac{\delta_j}{\gamma_j Var[\frac{B_T}{B_t}\phi V_t + g_{swp}(T)]} \left(E[g_\perp(T)] - \gamma_j \frac{\pi_j}{S_0^j} Cov[S_T^j, g_\perp(T)] \right) K \end{aligned} \quad (4.3)$$

where $g_Y(T)$ and $g_\perp(T)$ are defined by (2.4).

4.1.2 Market Equilibrium

We derive the equilibrium price and volume of the swap contract. From the equilibrium condition, the equilibrium price K^* is solved by

$$k_l^*(K;\phi) = k_s^*(K;\phi).$$

Substituting (4.3) into the equilibrium condition and solving with respect to K, we have an equilibrium swap price,

$$\begin{aligned} K^* = \frac{1}{E[g_\perp(T)] - \frac{\gamma_l\gamma_s}{\gamma_l+\gamma_s}\left(\frac{\pi_l}{S_0^l}Cov[S_T^l, g_\perp(T)] + \frac{\pi_s}{S_0^s}Cov[S_T^s, g_\perp(T)]\right)} \Bigg[E[g_Y(T)] + \frac{B_T}{B_t}\phi E[V_t] \\ - \frac{\gamma_l\gamma_s}{\gamma_l+\gamma_s}\Bigg\{ \frac{\pi_l}{S_0^l}\left(\frac{B_T}{B_t}\phi Cov\left[S_T^l, V_t\right] + Cov[S_T^l, g_Y(T)]\right) \\ + \frac{\pi_s}{S_0^s}\left(\frac{B_T}{B_t}\phi Cov\left[S_T^s, V_t\right] + Cov[S_T^s, g_Y(T)]\right)\Bigg\}\Bigg]. \quad (4.4) \end{aligned}$$

The equilibrium volume is also obtained by substituting (4.4) into (4.3). However, we are not available to have an explicit formula of the equilibrium volume.

5 Numerical Results

We have derived market equilibriums for both unilateral and bilateral counterparty risk models in the above. We however have never obtained closed formulae about the equilibrium prices and volumes. Especially, the equilibrium volume of the bilateral risk case has never even been in explicit form. Specifying the market model thus enables us to calculate the price and volume and to analyze the effect of the collateral on the equilibrium in a numerical way. We consider the two-period financial market with a finite probability space, and set $t = 0, \frac{1}{2}T, T$.

Let us introduce the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. There are four states in the one-period economy, we denote the state i by ω_i $(i = 1, \ldots, 4)$, i.e., $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ and $\mathcal{F} = 2^\Omega$ (c.f., Musiela and Zariphopoulou (2004)). The state ω_i is assumed to be assigned to the probability measure such that

$$P_i := \mathbb{P}(\omega_i) \quad i = 1, 2, \ldots, 4.$$

Under the probability space, the risky business values S^j of both participants j and the nontraded asset Y vary upward or downward during a period according to the economic state. The variation of S^j $(j = l, s)$ for a period is defined as

$$\Delta S^l(\omega_i) = \frac{S_{t+\Delta t}^l}{S_t^l} = \begin{cases} u_l, & i = 1, 2, \\ d_l, & i = 3, 4, \end{cases} \quad \Delta S^s(\omega_i) = \frac{S_{t+\Delta t}^s}{S_t^s} = \begin{cases} u_s, & i = 1, 4, \\ d_s, & i = 2, 3, \end{cases}$$

where $\Delta t = \frac{1}{2}T$ and

$$u_j = e^{\sigma_j\sqrt{\Delta t}}, \quad d_j = e^{-\sigma_j\sqrt{\Delta t}}$$

for $j = l, s$. On the other hand, the variation of Y is defined as

$$\Delta Y(\omega_i) = \frac{Y_{t+\Delta t}^l}{Y_t^l} = \begin{cases} u_Y, & i = 1, 3, \\ d_Y, & i = 2, 4 \end{cases}$$

where

$$u_Y = e^{\sigma_Y \sqrt{\Delta t}}, \quad d_Y = e^{-\sigma_Y \sqrt{\Delta t}}.$$

Note that, in this set up, there are different changes for S^j and Y at same states. This makes the financial market model incomplete, in which the agents do not necessarily accomplish the perfect hedge of the derivative.

Next, we respectively model the default events and the recovery rate. We use so-called constructed model to model the default event, that is, there exists threshold $L_j(> 0)$ $(j = l, s)$, the agent fails to payout the claim if the terminal business value S_T^j is less than L_j. We assume $S_0^j d_j^2 < L_j$ $(j = l, s)$. The default indicator function is then represented by

$$1_{D_j} = 1_{S_T^j < L_j}$$

for $j = l, s$. As introduced by Henderson and Liang (2014), we suppose that the recovery rate for the agent j's payment is determined as

$$\eta(S_T^j) = \eta_j \frac{S_T^j}{L_j}$$

where $\eta_j \in [0, 1]$ is a constant for $j = l, s$.

Now we note the relation between the counterparty risk and the exposure of the claim. The model defined in the above can describe the wrong-way risk too. To show that, let the economy is driven ω_3 at $t = \frac{1}{2}T$ and ω_3 at $t = T$ and consider the option contract. Under this path, $S_T^s = S_0^s d_s^2$ and $Y_0 u_Y^2$ at the maturity. Therefore, the payoff of the is maximum while the seller of it defaults. Increasing of P_3 hence makes the wrong-way risk more severe in this example.

In the final, we provide the risk-neutral martingale measure $\mathbb{Q}$ to estimate MtM value of the derivative contracts in the bellows. We identify the measure in a naive manner, although there are many discussions to give it because of the incomplete market model. Supposing that the underlying asset Y of the claim is traded in the financial market, we set the martingale measure to make it martingale, i.e., the martingale measure $\mathbb{Q}$ satisfies that

$$\begin{cases} \sum_{i=1}^{4} Q_i Y_{\Delta t}(\omega_i) = B_{\Delta t} Y_0 \\ \sum_{i=1}^{4} Q_i = 1 \end{cases} \tag{5.1}$$

where $Q_i := \mathbb{Q}(\omega_i)$ $(i = 1, 2, \ldots, 4)$. Solving (5.1) gives us the marginal probability as

$$Q := Q_1 + Q_3 = \frac{B_{\Delta t} - d_Y}{u_Y - d_Y}. \tag{5.2}$$

Note that we are not able to individually identify Q_i under the condition (5.1) only. It is, however, sufficient for the calculation of MtM value to have the marginal probability Q in (5.2). We denote the expectation operator $E^Q[\cdot]$ under the measure $\mathbb{Q}$.

5.1 Unilateral Counterparty Risk Case

In this section, we examine the market equilibrium for the option contract. We first derive the MtM value and provide the numerical results.

ϕ	$P_3 = 0.05$ Volume	$P_3 = 0.05$ Price	$P_3 = 0.45$ Volume	$P_3 = 0.45$ Price
0%	48.46	11.49	0.00	7.03
10%	46.76	11.76	0.00	7.68
20%	44.90	12.02	0.00	8.32
30%	42.90	12.29	0.00	8.96
40%	40.76	12.56	1.54	9.60
50%	38.49	12.83	21.82	10.24
60%	36.09	13.10	39.37	10.88
70%	33.59	13.37	53.92	11.52
80%	30.99	13.64	65.51	12.17
90%	28.32	13.91	74.38	12.81
100%	25.59	14.18	80.90	13.45
110%	22.82	14.45	85.45	14.09
120%	20.03	14.72	88.40	14.73
130%	17.24	14.99	90.10	15.37
140%	14.46	15.26	90.81	16.01
150%	11.72	15.53	90.78	16.66
160%	9.02	15.80	90.18	17.30
170%	6.39	16.07	89.17	17.94
180%	3.83	16.34	87.86	18.58
190%	1.36	16.61	86.33	19.22
200%	0.00	16.88	84.67	19.86

Table 1: Equilibrium prices and volumes for the option.

5.1.1 Building Blocks

At first, the net option payoff with the couterparty risk and the collateral agreement is represented by

$$g_{opt}(T) = (H(T) - B_T^c \phi V_0)(1 - 1_{S_T^s \geq L_s}) + \eta_s \frac{S_T^s}{L_s} H(T) 1_{S_T^s < L_s}. \tag{5.3}$$

Next we derive the MtM value. The MtM value V under the martingale measure $\mathbb{Q}$ in (5.2) is given by

$$V_0 = B_T^{-1} E^Q[(Y_T - K)^+] = e^{-rT}\{Q^2 H_{++} + 2Q(1-Q)H_+\}$$

where K is the strike price s.t. $K > Y_0 d_Y^2$, and we denote $H_{++} = Y_0 u_Y^2 - K$ and $H_+ = Y_0 u_Y d_Y - K$. Finally, the equilibrium pricing formula used in this implementation with (5.3) is given in Section 3.1.2. Related terms in the formulae of the equilibrium price and volume are given in Appendix A.

5.1.2 Numerical Results

We implement the sensitivity analysis of the market equilibrium for the option contract with respect to the coverage ratio, and we numerically investigate the influences of the collateral agreement on the equilibrium price and volume.

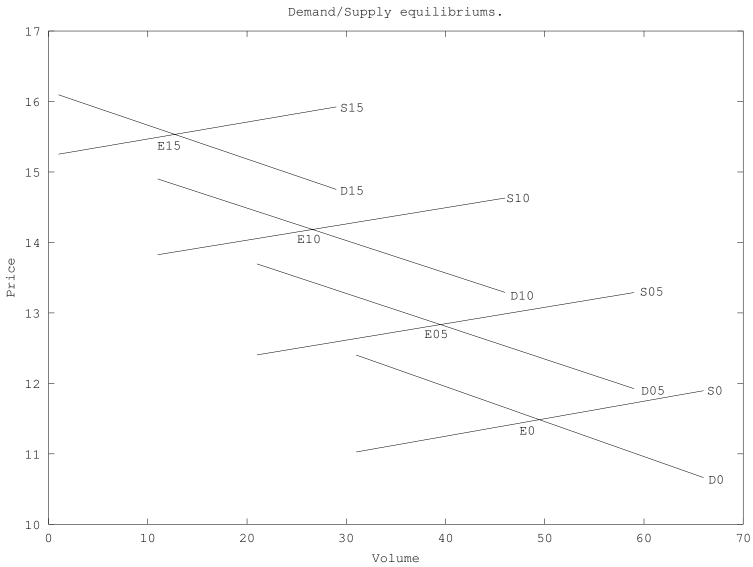

Figure 1: Market equilibriums of the option contract when the buyer has the right-way risk. The demand/supply curves are plotted for $\phi = 0\%, 50\%, 100\%, 150\%$ (respectively denotes "0", "05", "10" and "15" in the graph) and $P_3 = 0.05$, the notations "D", "S" and "E", respectively, mean the demand/supply curve and its equilibrium point.

The parameters used in this examination are as follows: $P_2 = P_4 = 0.25$, $S_0^l = 100.0$, $\sigma_l = 0.1$, $\gamma_l = 0.0002$, $S_0^s = 100.0$, $\sigma_s = 0.4$, $\gamma_s = 0.0001$, $Y_0 = 100.0$, $\sigma_Y = 0.2$, $K = 90.0$, $\eta_s = 0.5$, $L_s = 90.0$, $r = r_c = 0.05$ and $T = 1.0$. The remaining parameters P_1 and P_3 are changed to describe the right-way/wrong-way risk, such that

$$P_1 = 1 - (P_2 + P_3 + P_4) \tag{5.4}$$

where P_2 and P_4 are fixed. For the option contract, we suppose to measure the right-way/wrong-way risk by the correlation coefficient between $H(T)$ and $1_{S_T^s < L_s}$, i.e.,

$$WR := \frac{Cov[H(T), 1_{S_T^s < L_s}]}{\sqrt{Var[H(T)]Var[1_{S_T^s < L_s}]}}.$$

The sign of WR especially indicates whether the buyer of the option has the right-way/wrong-way risk in the option contract. The buyer has the right way risk about the seller's default if $WR < 0$, and she/he has the wrong way risk if $WR > 0$. The combination of P_1 and P_3 varies the state of the right-way/wrong-way risk.

Table 1 shows the equilibrium volume and price for $P_3 = 0.05, 0.45$ and each coverage ratio ϕ. The buyer of the option has the right-way risk when $P_3 = 0.05$ since $WR = -0.23$ at $P_3 = 0.05$, and she/he has the wrong-way risk when $P_3 = 0.45$ since $WR = 0.40$ at $P_3 = 0.45$. We first argue about the result of the volume. For the case of the right-way risk (i.e., the case of $P_3 = 0.05$), the equilibrium volume monotonically decreases with increasing of the coverage ratio ϕ from the

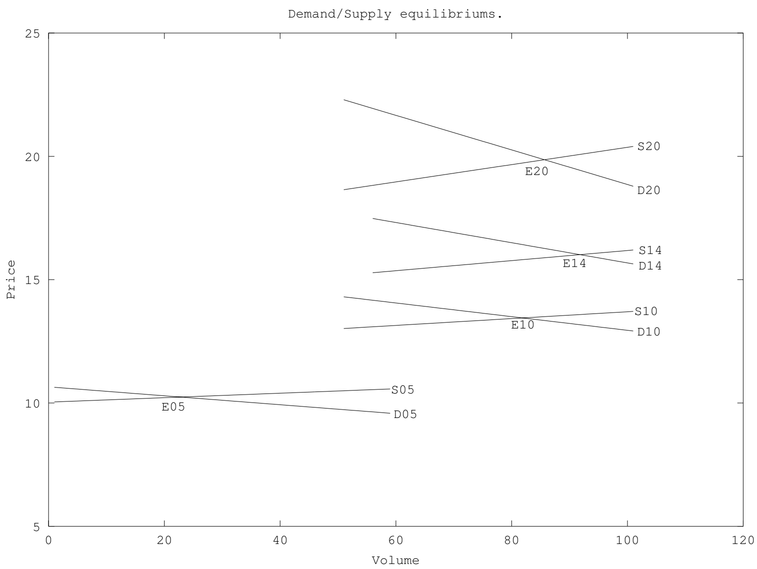

Figure 2: Market equilibriums of the option contract when the buyer has the wrong-way risk. The demand/supply curves are plotted for $\phi = 50\%, 100\%, 140\%, 200\%$ (respectively denotes "05", "10", "14" and "20" in the graph) and $P_3 = 0.45$, the notations "D", "S" and "E", espectively, mean the demand/supply curve and its equilibrium point.

table. That is, the more requirement of the collateral decreases the liquidity of it. On the other hand, at the case of the wrong-way risk (i.e., the case of $P_3 = 0.45$), the equilibrium volume is a few or zero for small coverage ratios. In fact, the volume is zero for $0\% \leq \phi \leq 30\%$ at $P_3 = 0.45$. However, the amount of the volume increases and decreases after achieving the maximum volume with increasing of the coverage ratio. We note that this characteristic is shown in Section 3.1.3. That is, the collateralization is able to recover the option trading although the market participant faces the wrong-way risk.

Next, we report the numerical result about the equilibrium price. The equilibrium price monotonically increases with increasing of the coverage ratio for both P_3's. This is supported on the theoretical point of view. Since we set that the risk-free rate r equals to the collateral rate r_c, we have $\partial p^*/\partial \phi \geq 0$ from Section 3.1.3. We also have an interest in the case of $r_c > r$. In the previous section, we pointed out that the equilibrium price decreases with the increase of the coverage ratio when $r_c = 0.1$ and $r = 0.01$. Table 2 shows the numerical results for the equilibrium price, and the results show that the price monotonically increase with the increase of the coverage ratio for each P_3.

Finally, we observe the changes of demand/supply curves when the coverage ratio increases. We have obtained the results that the influence of the collateral on the traded volume differs depending on the state of the right-way/wrong-way risk. The implementation verifies the reason of this difference. Figure 1, 2 show demand/supply curves for the option contract under the right-way/wrong-way risk, respectively. Figure 1 corresponds to the case of $WR = -0.23$ (or $P_3 = 0.05$) and the demand/supply curve are described for the case of $\phi = 0\%, 50\%, 100\%, 150\%$. The figure

ϕ \ P_3	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
0%	7.89	10.52	11.96	12.64	12.80	12.55	11.95	10.99	9.61	7.64	4.74
10%	7.96	10.64	12.13	12.87	13.07	12.87	12.31	11.40	10.06	8.14	5.29
20%	8.02	10.76	12.31	13.09	13.34	13.19	12.67	11.80	10.50	8.63	5.85
30%	8.09	10.89	12.49	13.32	13.62	13.51	13.03	12.20	10.95	9.13	6.41
40%	8.15	11.01	12.66	13.55	13.89	13.82	13.40	12.61	11.40	9.62	6.97
50%	8.22	11.13	12.84	13.77	14.17	14.14	13.76	13.01	11.84	10.12	7.53
60%	8.28	11.25	13.02	14.00	14.44	14.46	14.12	13.41	12.29	10.61	8.08
70%	8.35	11.38	13.19	14.23	14.72	14.78	14.48	13.82	12.74	11.11	8.64
80%	8.41	11.50	13.37	14.46	14.99	15.10	14.84	14.22	13.18	11.60	9.20
90%	8.48	11.62	13.54	14.68	15.27	15.42	15.20	14.62	13.63	12.10	9.76
100%	8.54	11.74	13.72	14.91	15.54	15.74	15.57	15.03	14.08	12.59	10.31
110%	8.61	11.87	13.90	15.14	15.82	16.06	15.93	15.43	14.52	13.09	10.87
120%	8.67	11.99	14.07	15.36	16.09	16.38	16.29	15.83	14.97	13.58	11.43
130%	8.74	12.11	14.25	15.59	16.36	16.70	16.65	16.24	15.41	14.08	11.99
140%	8.80	12.23	14.43	15.82	16.64	17.02	17.01	16.64	15.86	14.57	12.55
150%	8.87	12.36	14.60	16.04	16.91	17.34	17.37	17.04	16.31	15.06	13.10
160%	8.94	12.48	14.78	16.27	17.19	17.66	17.74	17.45	16.75	15.56	13.66
170%	9.00	12.60	14.96	16.50	17.46	17.97	18.10	17.85	17.20	16.05	14.22
180%	9.07	12.72	15.13	16.73	17.74	18.29	18.46	18.25	17.65	16.55	14.78
190%	9.13	12.85	15.31	16.95	18.01	18.61	18.82	18.66	18.09	17.04	15.34
200%	9.20	12.97	15.49	17.18	18.29	18.93	19.18	19.06	18.54	17.54	15.89

Table 2: Equilibrium prices for the option when $r_c > r$.

shows that the increase of the coverage ratio shifts the demand curve to upward, which means the increase of the demand and the supply curve to upward which shows the decrease of the supply. Furthermore, the decrease of the supply is larger than the increase of the demand. This leads the decrease of the traded volume and the increase of the price. These phenomena are interpreted as follows: The buyer is going to feel the safety of the derivative contract with increasing of the collateral, and this leads the increase of the demand. The seller, on the other hand, feels not good to grow up the temporal payment associated with the collateralization, and she/he also has to pay much money of the collateral for the increase of the coverage ratio. Another says, the seller loses the opportunity to optimally invest in other assets. This reduces the incentive to trade more claim contracts and shifts the supply curve to upward when the coverage ratio increases. Next, we consider the case of the wrong-way risk. Figure 2 corresponds to the case of $WR = 0.40$ (or $P_3 = 0.45$) and the demand/supply curve are described for the case of $\phi = 50\%, 100\%, 140\%, 200\%$ (note that the volume is maximum at $\phi = 140\%$). As the right-way risk case, the figure shows that the increase of the coverage ratio shifts the demand curve to upward and the supply curve to upward. The different point is that the decrease of the supply is smaller than the increase of the demand up to $\phi = 140\%$. This gives the increase of the traded volume (E05 $\rightarrow$ E10 $\rightarrow$ E14). After that, the traded volume is contracted with increasing of the coverage ratio (E14 $\rightarrow$ E20). That is, the decrease of the supply is larger than the increase of the demand. Adding this, the slopes of the demand curve significantly becomes more sharper. These lead reductions of the traded volume.

P_4	0.00	0.15	0.65
WR_l	-0.34 (R)	-0.20 (R)	0.03 (W)
WR_s	0.02 (R)	-0.20 (W)	-0.32 (W)

Table 3: Right-way/Wrong-way risks for the couterparties. (R) ((W)) denotes the right-way (wrong-way) risk.

How do we interpret these trends? At first, as to the increase of the demand, the buyer is going to feel safety for the derivative contract with increasing of the collateral as the right-way risk case. Next, we consider the seller's behavior. The seller cannot withdraw any posted collateral if she/he defaults. The amount of posted collateral, however, is smaller than the highest money amounts of option payoff in our model, i.e.,

$$\phi V_0 < Y_0 u_Y^2 - K,$$

for $0\% \leq \phi \leq 200\%$. This implies that it is profitable for the seller to default when $Y_T = Y_0 u_Y^2$. Such a default likely arises since the market is in the wrong-way risk environment. This fact relatively softens the decrease of the supply with the increase of the collateral (E05 $\to$ E10 $\to$ E14). On the other hand, when $\phi = 200\%$, the difference between the collateral amount and the derivative payoff significantly reduces. This loses the advantage of the seller for her/his default, the decrease of the supply expands. This cuts back the liquidity (E14 $\to$ E20).

Let us summarize the above results. The equilibrium price of the option contract is significantly reflected by the collateral agreement. The price increases (decreases) when the collateral amount increases (decreases). This characteristic consists with the one of the risk-neutral pricing. The equilibrium volume of it is also affected by the collateralization. But, it is not uniform. That is, the collateralization recovers the liquidity of the claim while it reduces the liquidity of the claim. These trends are subject to the states of the right-way/wrong-way risk.

5.2 Bilateral Counterparty Risk Case

In this section, we implement the market equilibriums for the swap contract. We first derive the MtM value and provide the numerical results as the option case.

5.2.1 Building Blocks

The net swap payoff for the long-position with the couterparty risk and the collateral agreement is represented by

$$\begin{aligned} g_{swp}(T) = &\left(Y_T - \frac{B_T^c}{B_t^c}\phi V_t 1_{V_t \geq 0} \right) 1_{S_T^s \geq L_s} + \eta_s \frac{S_T^s}{L_s} Y_T 1_{S_T^s < L_s} \\ &- \left(K + \frac{B_T^c}{B_t^c}\phi V_t 1_{V_t < 0} \right) 1_{S_T^l \geq L_l} - \eta_l \frac{S_T^l}{L_l} K 1_{S_T^l < L_l}. \end{aligned} \tag{5.5}$$

We rewrite (5.5) as

$$g_{swp}(T) = g_Y(T) - K g_{\perp}(T)$$

ϕ	$P_4 = 0.00$ Volume	$P_4 = 0.00$ Price	$P_4 = 0.15$ Volume	$P_4 = 0.15$ Price	$P_4 = 0.65$ Volume	$P_4 = 0.65$ Price
0%	68.54	106.78	50.26	102.36	0.00	88.97
10%	67.64	106.84	49.44	102.34	0.00	88.82
20%	66.85	106.91	48.70	102.32	0.00	88.67
30%	66.16	106.97	48.05	102.29	0.00	88.52
40%	65.58	107.03	47.48	102.27	0.00	88.37
50%	65.08	107.10	46.98	102.24	0.00	88.22
60%	64.67	107.16	46.56	102.22	0.00	88.07
70%	64.34	107.22	46.21	102.19	0.00	87.92
80%	64.10	107.29	45.93	102.17	0.00	87.77
90%	63.93	107.35	45.72	102.14	0.19	87.62
100%	63.83	107.41	45.57	102.12	1.77	87.47
110%	63.81	107.48	45.50	102.09	3.45	87.31
120%	63.86	107.54	45.49	102.07	5.23	87.16
130%	63.98	107.60	45.56	102.05	7.10	87.01
140%	64.17	107.67	45.70	102.02	9.07	86.86
150%	64.44	107.73	45.92	102.00	11.13	86.71
160%	64.78	107.79	46.22	101.97	13.28	86.56
170%	65.20	107.86	46.61	101.95	15.51	86.41
180%	65.69	107.92	47.09	101.92	17.83	86.26
190%	66.27	107.98	47.67	101.90	20.21	86.11
200%	66.93	108.05	48.36	101.87	22.66	85.96

Table 4: Equilibrium volumes and prices for the swap contract.

where

$$g_Y(T) = Y_T \left\{ 1 - \left(1 - \eta_s \frac{S_T^s}{L_s} \right) 1_{S_T^s < L_s} \right\} - \frac{B_T^c}{B_t^c} \phi V_t \{ 1_{V_t \geq 0, S_T^s \geq L_s} + 1_{V_t < 0, S_T^l \geq L_l} \},$$

$$g_\perp(T) = 1 - \left(1 - \eta_l \frac{S_T^l}{L_l} \right) 1_{S_T^l < L_l}.$$

Recall, the pricing for the MtM is independently done with our equilibrium scheme and we set t as the date of MtM. By the standard approach, the risk-neutral swap price $\tilde{K}$ at the contract date solves to

$$E^Q[B_T^{-1}(Y_T - K)] = 0.$$

Since the risk-free rate r is a constant, the swap price is then given by

$$\tilde{K} = E^Q[Y_T] = Y_0 \{ Q u_Y + (1 - Q) d_Y \}^2 \tag{5.6}$$

where Q is defined in (5.2). The MtM is assumed to be computed and executed by the formula

$$V_t = E_t^Q \left[\frac{B_t}{B_T} (Y_T - \tilde{K}) \right].$$

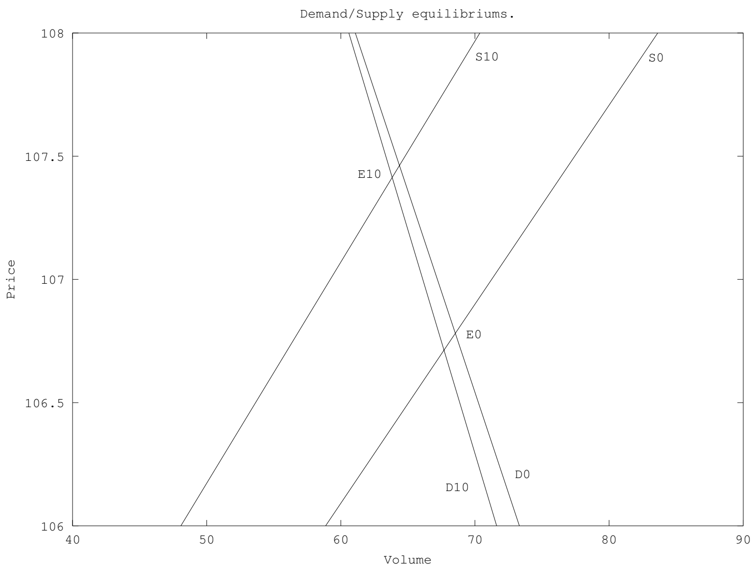

Figure 3: Market equilibriums of the option contract when the buyer has the right-way risk. The demand/supply curves are plotted for $\phi = 0\%, 100\%$ (respectively denotes “0”, “10” in the graph) and $P_4 = 0.00$, the notations “D”, “S” and “E”, respectively, mean the demand/supply curve and its equilibrium point.

For convenience, we denote

$$\begin{aligned} V_t^u &:= V_t(\{\omega_1, \omega_3\}) = (1-Q)(u_Y - d_Y)Y_0\{Qu_Y + (1-Q)d_Y\} - \tilde{K}, \\ V_t^d &:= V_t(\{\omega_2, \omega_4\}) = Q(d_Y - u_Y)Y_0\{Qu_Y + (1-Q)d_Y\} - \tilde{K}. \end{aligned}$$

The equilibrium pricing formula used in this implementation with (5.5) is given in Section 4.1.2. Corresponding terms in the formulae of the equilibrium swap rate and volume are given in Appendix B.

5.2.2 Numerical Results

We examine the sensitivity analysis of the market equilibrium of the swap contract with respect to the coverage ratio, and we numerically investigate the influences of the collateral agreement on the equilibrium swap rate and volume.

The parameters used in this examination are as follows: $P_2 = 0.15$, $P_3 = 0.05$, $S_0^l = 100.0$, $\sigma_l = 0.1$, $\gamma_l = 0.0002$, $S_0^s = 100.0$, $\sigma_s = 0.4$, $\gamma_s = 0.0001$, $Y_0 = 100.0$, $\sigma_Y = 0.2$, $\eta_l = 0.5$, $L_l = 90.0$, $\eta_s = 0.5$, $L_s = 90.0$, $r = r_c = 0.05$ and $T = 1.0$. We set the date of MtM as $t = \frac{1}{2}T$, i.e., $t = 0.5$. The remaining parameters P_1 and P_4 are changed to describe the right-way/wrong-way risk, such that

$$P_1 = 1 - (P_2 + P_3 + P_4) \tag{5.7}$$

where P_2 and P_3 are fixed. For the option contract, we suppose to measure the right-way/wrong-way risks for both participants by the correlation coefficient between V_t and $1_{S_T^j < L_j}$ $(j = l, s)$,

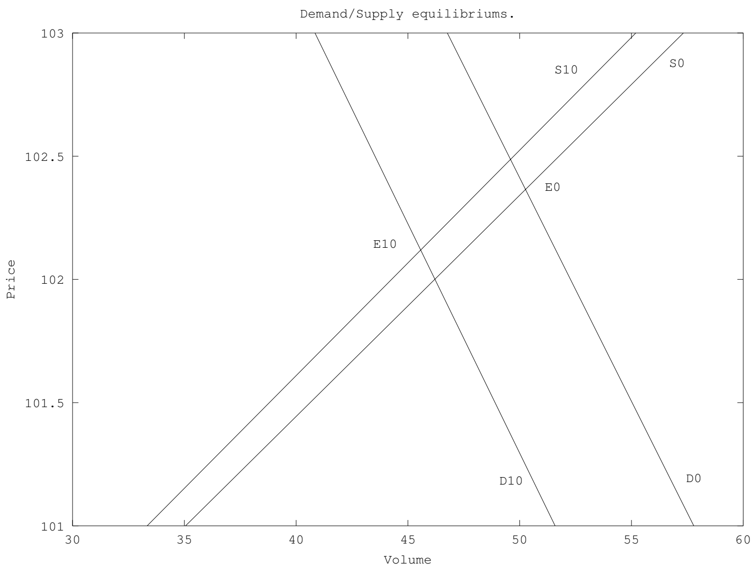

Figure 4: Market equilibriums of the option contract when the buyer has the wrong-way risk. The demand/supply curves are plotted for $\phi = 0\%, 100\%$ (respectively denotes "0", "10" in the graph) and $P_4 = 0.15$, the notations "D", "S" and "E", respectively, mean the demand/supply curve and its equilibrium point.

i.e.,

$$WR_l := \frac{Cov[V_t, 1_{S_T^s < L_s}]}{\sqrt{Var[V_t]Var[1_{S_T^s < L_s}]}}, \quad WR_s := \frac{Cov[V_t, 1_{S_T^l < L_l}]}{\sqrt{Var[V_t]Var[1_{S_T^l < L_l}]}}.$$

As the unilateral case, the negative sign of WR_j $(j = l, s)$ means whether the counterparties have the right-way/wrong-way risk in the swap contract. The buyer has the right-way risk about the seller's default if $WR_l < 0$, and she/he has the wrong-way risk if $WR_l > 0$. The seller is posed the right-way risk about the buyer's default if $WR_s > 0$, and she/he is posed the wrong-way risk if $WR_s < 0$.

The numerical results for the equilibrium volume and price are shown in Table 4, and it is described for $P_4 = 0.00, 0.15, 0.65$. P_4 determines the state of the right-way/wrong-way risk for the both agents as described in Table 3. From Table 3, the buyer and seller are in the right-way risk at $P_4 = 0.00$, the buyer is at the right-way risk and the seller is in the wrong-way risk at $P_4 = 0.15$, and they are in the wrong-way risk at $P_4 = 0.65$.

We first argue about the impact of the collateralization on the equilibrium swap rate. For each P_4, the swap rate monotonically changes according to the coverage ratio ϕ. Whether the monotone increasing or decreasing in the swap rate depends on the level of P_4 or the degree of the right-way risk for both participants as mentioned in Section 2.3.2, i.e., the swap rate monotonically increases with the increase of the collateral agreement if the degree of the right-way risk of the short-holder is bigger than the one of the long-holder, and monotonically decreases otherwise. From Table 3, the degree of the right-way risk of the long-holder is stronger than the one of the short-holder for $P_4 = 0.15, 0.65$. The swap rate has a monotone a monotone decreasing for $P_4 = 0.15, 0.65$. Thus,

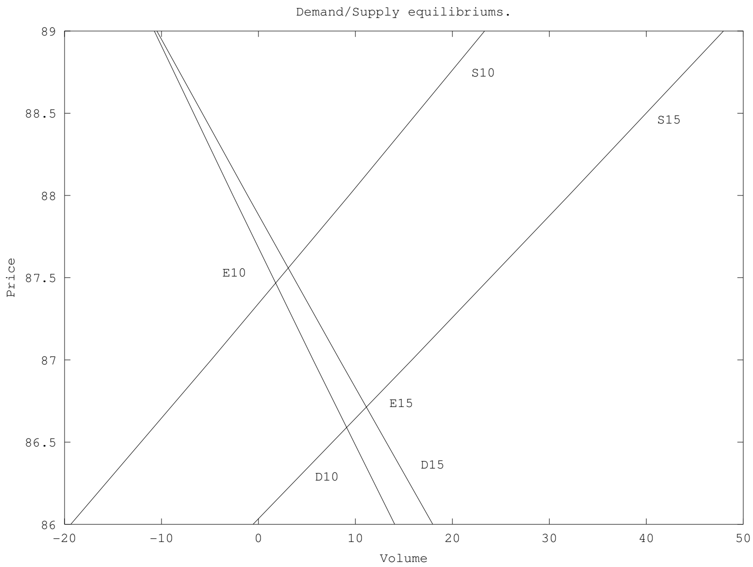

Figure 5: Market equilibriums of the option contract when the buyer has the wrong-way risk. The demand/supply curves are plotted for $\phi = 100\%, 150\%$ (respectively denotes "10", "15" in the graph) and $P_4 = 0.65$, the notations "D", "S" and "E", respectively, mean the demand/supply curve and its equilibrium point.

at $P_4 = 0.15, 0.65$, the sensitivity of the swap rate by our pricing rule coincides with the one by the standard approach given in Section 2.3.2. The influence on the swap rate, however, is not significant in contrast to the option contract. For example, at $P_4 = 0.00$, the swap rate is 107.41 with 100% collateralization while the trade is done at a price of 106.78 with no collateral. This aspect is also observed other P_4 cases. Thus, our model shows that the impact of the collateral agreement on the swap rate is limited.

We next review the result of the volume. At first, the relation between the coverage ratio and the volume is not monotonicity. We have the minimum volumes for $P_4 = 0.00, 0.15$, the volume is minimum at $\phi = 110\%, 120\%$ for $P_4 = 0.00, 0.15$, respectively. However, the differences between the minimum volume and other volumes are not significant in contrast to the option case. In fact, the minimum volume is 63.81 and the maximum volume is 68.54 at $P_4 = 0.00$. Next, the table shows that the collateral recovers the trade of the swap contract. The case of $P_4 = 0.65$ corresponds that the both counterparties have the wrong-way risk in the swap contract. In this case, the transaction of the claim is not achieved with small collateral since the equilibrium volumes are zero for $0\% \leq \phi \leq 80\%$. The volume, however, is positive and increases with the increase of the coverage ratio ϕ for $\phi \geq 90\%$. This characteristic is also observed in the option case ($P_3 = 0.45$). Our implementation thus shows that the collateral is partially effective on the volume under the both counterparties have the wrong-way risk while the effect of it on the volume is not significant in some counterpaty risk cases.

Finally, we observe the shifts of demand/supply curves. Figures 3-5 describe the changes of demand/supply curves for each P_4. Figure 3 plots for $\phi = 0\%, 100\%$ at $P_4 = 0.00$. Figure 4 plots for $\phi = 0\%, 100\%$ at $P_4 = 0.15$. Both figures show that the equilibrium volume declines with the increase of the coverage ratio ϕ (E0 $\rightarrow$ E10 in both figures). From figures, we observe that the volume is reduced as a result of both curves shift to the left side or the slopes of both curves change to decrease the volume. Recall that, there are possibilities to post the collateral for both counterparties in the swap contract. The shifts of both curves to the left hand side, are interpreted as the result of reflecting the possibility of posting collateral. Figure 5 plots for $\phi = 100\%, 150\%$ at $P_4 = 0.65$. This figure shows that the equilibrium volume is increased with the increase of the coverage ratio ϕ (E10 $\rightarrow$ E15). From the figure, we observe that the volume increases as a result of both curves shift to the right side or the slopes of both curves change to increase the volume. This trend is interpreted as follows: As mentioned above, the trade is not entered for small coverage ratio. However, the agents are going to feel easy to deal the derivative contract according to the increase of collateral amount.

Let us summarize numerical results for the swap contract case. The equilibrium swap rate is slightly changed with changes in the collateral. It also seems that the swap rate is not significantly affected by the collateral agreement compared with the option price case. The swap rate monotonically changes with change in the collateral amount according to the right-way/wrong-way risk for the agents. In particular, the sensitivity of the swap rate in our equilibrium pricing partially coincides with the one in the standard pricing approach. As to the equilibrium volume, the impact of the collateral is not significant for which either counterparty has the right-way risk at least. However, the both counterparties become to have a positive feeling with the increase of the collateral in the case of both agents are posed the wrong-way risk, this makes the agents to enter more swap contract and recover the transaction of it. Therefore, it is the benefit for the swap contract to introduce the collateralization in the case of worst counterparty risk as observed in the option case.

6 Summary

In this work, we have demonstrated the effects of the collateral agreement on the derivative transactions in Microeconomics point of view. The results are mainly obtained by the numerical way. We considered the option contract as an example of the unilateral counterparty risk and the swap contract as an example of bilateral counterparty risk. We also took into account the right-way/wrong-way risk by using the so-called structural model in the classical credit risk modeling.

Our model showed that the differences of the impacts by the unilateral/bilateral counterparty risk case. For the former case, the collateral agreement affects both liquidity and price for the derivative contract. We observed that the price monotonically increases with an increase of the collateral amount as shown in the standard pricing approach (i.e., risk-neutral pricing). On the other hand, the effect on the liquidity is not uniform. The collateralization has an aspect to increase the liquidity even if the buyer has the wrong-way risk while it decreases the liquidity otherwise. As in the latter case, the effects of the collateral agreement on the liquidity and the price are limited compared with the unilateral case. The price especially is not effective with the collateral agreement. On the other hand, under the case that the both counterparties are in the wrong-way risk, the collateralization grows up the liquidity for large collateral amount. Therefore, for both the unilateral/bilateral cases, the collateralization has a possible to recover the liquidity even if either or both market participants have the wrong-way risk for the counterparty's default.

It goes without saying that the collateralization is expected to reduce the counterparty risk in the derivative contracts. What is lost by the collateralization? Our study verified it by considering an equilibrium analysis for the derivative contracts. The transaction opportunities with unilateral counterparty risk like an option contract might be reduced by the collateralization. We don't have to worry about losing opportunities to trade the swap with the collateralization. G20 in Sep. 2013 decided to impose the collateral agreements in the interest rate swap contracts. Our result thus might support their decision.

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A Building Blocks for Unilateral Counterparty Risk Case

In this appendix, we provide the building blocks to calculate the equilibrium price and volume for the option contract (which corresponds to the numerical examination in Section 5.1).

The expectation, variance of g_{opt} and the covariance of S_T^j and g_{opt} are then calculated as follows. At first, the expectation is

$$\begin{aligned}E[g_{opt}(T)] =& P_1\{(P_1+P_3)(H_{++}-B_T^cC(\phi))+(P_2+P_4)(H_+-B_T^cC(\phi))\}\\ &+P_2\left\{P_1(H_+-B_T^cC(\phi))+P_3\eta_s\frac{S_0^sd_s^2}{L_s}H_+-P_4B_T^cC(\phi)\right\}\\ &+P_3\left\{P_1(H_{++}-B_T^cC(\phi))+P_2\eta_s\frac{S_0^sd_s^2}{L_s}H_++P_3\eta_s\frac{S_0^sd_s^2}{L_s}H_{++}+P_4(H_+-B_T^cC(\phi))\right\}\\ &+P_4\{(P_1+P_3)(H_+-B_T^cC(\phi))-(P_2+P_4)B_T^cC(\phi)\}.\end{aligned}$$

The variance is

$$\begin{aligned}Var[g_{opt}(T)] &= E[(g_{opt}(T))^2]-E[g_{opt}(T)]^2\\ &= P_1\{(P_1+P_3)(H_{++}-B_T^cC(\phi))^2+(P_2+P_4)(H_+-B_T^cC(\phi))^2\}\\ &+P_2\left\{P_1(H_+-B_T^cC(\phi))^2+P_3\left(\eta_s\frac{S_0^sd_s^2}{L_s}H_+\right)^2+P_4(B_T^cC(\phi))^2\right\}\\ &\quad+P_3\left\{P_1(H_{++}-B_T^cC(\phi))^2+P_2\left(\eta_s\frac{S_0^sd_s^2}{L_s}H_+\right)^2\right.\\ &\quad\left.+P_3\left(\eta_s\frac{S_0^sd_s^2}{L_s}H_{++}\right)^2+P_4(H_+-B_T^cC(\phi))^2\right\}\\ &+P_4\{(P_1+P_3)(H_+-B_T^cC(\phi))^2+(P_2+P_4)(B_T^cC(\phi))^2\}-E[g_{opt}(T)]^2. \quad (A.1)\end{aligned}$$

Finally, the covariance is

$$\begin{aligned}&Cov[S_T^l,g_{opt}(T)] = E[S_T^lg_{opt}(T)]-E[S_T^l]E[g_{opt}(T)]\\ &= P_1\{P_1S_0^lu_l^2(H_{++}-B_T^cC(\phi))+P_2S_0^lu_l^2(H_+-B_T^cC(\phi))+P_3S_0^lu_ld_l(H_{++}-B_T^cC(\phi))+P_4S_0^lu_ld_l(H_+-B_T^cC(\phi))\}\\ &\quad+P_2\left\{P_1S_0^lu_l^2(H_+-B_T^cC(\phi))+P_3S_0^lu_ld_l\eta_s\frac{S_0^sd_s^2}{L_s}H_+-P_4S_0^lu_ld_lB_T^cC(\phi))\right\}\\ &\quad+P_3\left\{P_1S_0^ld_lu_l(H_{++}-B_T^cC(\phi))+P_2S_0^ld_lu_l\eta_s\frac{S_0^sd_s^2}{L_s}H_+\right.\\ &\quad\left.+P_3S_0^ld_l^2\eta_s\frac{S_0^sd_s^2}{L_s}H_{++}+P_4S_0^ld_l^2(H_+-B_T^cC(\phi))\right\}\\ &\quad+P_4\{(P_1S_0^ld_lu_l+P_3S_0^ld_l^2)(H_+-B_T^cC(\phi))-(P_2S_0^ld_lu_l+P_4S_0^ld_l^2)B_T^cC(\phi)\}\\ &\quad-E[S_T^l]E[g_{opt}(T)],\end{aligned}$$

$$
\begin{aligned}
&Cov[S_T^s, g_{opt}(T)] = E[S_T^s g_{opt}(T)] - E[S_T^s]E[g_{opt}(T)] \\
&= P_1\{P_1 S_0^s u_s^2 (H_{++} - B_T^c C(\phi)) + P_2 S_0^s u_s d_s (H_+ - B_T^c C(\phi)) + P_3 S_0^s u_s d_s (H_{++} - B_T^c C(\phi)) + P_4 S_0^s u_s^2 (H_+ - B_T^c C(\phi))\} \\
&\quad + P_2\left\{P_1 S_0^s d_s u_s (H_+ - B_T^c C(\phi)) + P_3 S_0^s d_s^2 \eta_s \frac{S_0^s d_s^2}{L_s} H_+ - P_4 S_0^s d_s u_s B_T^c C(\phi)\right\} \\
&\quad + P_3\left\{P_1 S_0^s d_s u_s (H_{++} - B_T^c C(\phi)) + P_2 S_0^s d_s^2 \eta_s \frac{S_0^s d_s^2}{L_s} H_+ \right. \\
&\quad \left. + P_3 S_0^s d_s^2 \eta_s \frac{S_0^s d_s^2}{L_s} H_{++} + P_4 S_0^s d_s u_s (H_+ - B_T^c C(\phi))\right\} \\
&\quad + P_4\{(P_1 S_0^s u_s^2 + P_3 S_0^s u_s d_s)(H_+ - B_T^c C(\phi)) - (P_2 S_0^s u_s d_s + P_4 S_0^s u_s^2) B_T^c C(\phi)\} \\
&\quad - E[S_T^s]E[g_{opt}(T)].
\end{aligned}
$$

B Building Blocks for Bilateral Counterparty Risk Case

In this appendix, we provide the building blocks to calculate the equilibrium price and volume of the swap contract (which corresponds to the numerical examination in Section 5.2).

The expectation of g_{swp} is

$$
E[g_{swp}(T)] = E[g_Y(T)] - KE[g_\perp(T)]
$$

where

$$
\begin{aligned}
E[g_Y(T)] &= E[Y_T 1_{S_T^s \geq L_s}] + \frac{\eta_s}{L_s} E[S_T^s Y_T 1_{S_T^s < L_s}] - \frac{B_T^c}{B_t^c}\phi E[V_t 1_{V_t \geq 0, S_T^s \geq L_s}] - \frac{B_T^c}{B_t^c}\phi E[V_t 1_{V_t < 0, S_T^l > L_l}], \\
E[g_\perp(T)] &= 1 - (P_3 + P_4)^2 + \frac{\eta_l}{L_l}(P_3 + P_4)^2 S_0^l d_l^2,
\end{aligned}
$$

with

$$
\begin{aligned}
E[Y_T 1_{S_T^s \geq L_s}] &= P_1\{(P_1 + P_3) Y_0 u_Y^2 + (P_2 + P_4) Y_0 u_Y d_Y\} + P_2 (P_1 Y_0 d_Y u_Y + P_4 Y_0 d_Y^2) \\
&\quad + P_3 (P_1 Y_0 u_Y^2 + P_4 Y_0 u_Y d_Y) + P_4\{(P_1 + P_3) Y_0 d_Y u_Y + (P_2 + P_4) Y_0 d_Y^2\}, \\
E[S_T^s Y_T 1_{S_T^s < L_s}] &= S_0^s d_s^2 Y_0 (P_2 d_Y + P_3 u_Y)^2, \\
E[V_t 1_{V_t \geq 0, S_T^s \geq L_s}] &= (P_1 + P_3(P_1 + P_4)) V_t^u, \\
E[V_t 1_{V_t < 0, S_T^l \geq L_l}] &= (P_2 + P_4(P_1 + P_2)) V_t^d.
\end{aligned}
$$

Next, the variance of g_{swp} is

$$
Var[g_{swp}(T)] = Var[g_Y(T)] + K^2 Var[g_\perp(T)] - 2KCov[g_Y(T), g_\perp(T)]
$$

where

$$\begin{aligned}
Var[g_Y(T)] =& Var[Y_T 1_{S_T^s \geq L_s}] + \left(\frac{\eta_s}{L_s}\right)^2 Var[S_T^s Y_T 1_{S_T^s < L_s}] \\
&+ \left(\frac{B_T^c}{B_t^c}\phi\right)^2 Var\left[V_t 1_{V_t \geq 0, S_T^s \geq L_s}\right] + \left(\frac{B_T^c}{B_t^c}\phi\right)^2 Var[V_t 1_{V_t < 0, S_T^l \geq L_l}] \\
&- 2\frac{B_T^c}{B_t^c}\phi Cov[Y_T 1_{S_T^s \geq L_s}, V_t 1_{V_t \geq 0, S_T^s \geq L_s}] - 2\frac{B_T^c}{B_t^c}\phi Cov[Y_T 1_{S_T^s \geq L_s}, V_t 1_{V_t < 0, S_T^l \geq L_l}] \\
&- 2\frac{\eta_s}{L_s}\frac{B_T^c}{B_t^c}\phi Cov[S_T^s Y_T 1_{S_T^s < L_s}, V_t 1_{V_t < 0, S_T^l \geq L_l}], \\
Var[g_\perp(T)] =& Var\left[\left(1 - \eta_l \frac{S_T^l}{L_l}\right) 1_{S_T^l < L_l}\right] \\
=& (P_3 + P_4)^2 \left(1 - \eta_l \frac{S_0^l d_l^2}{L_l}\right)^2 - \left\{(P_3 + P_4)^2 \left(1 - \eta_l \frac{S_0^l d_l^2}{L_l}\right)\right\}^2, \\
Cov[g_Y(T), g_\perp(T)] =& E[g_Y(T) g_\perp(T)] - E[g_Y(T)]E[g_\perp(T)],
\end{aligned}$$

with

$$\begin{aligned}
Var[Y_T 1_{S_T^s \geq L_s}] =& P_1\{(P_1 + P_3)(Y_0 u_Y^2)^2 + (P_2 + P_4)(Y_0 u_Y d_Y)^2\} \\
&+ P_2\{P_1(Y_0 d_Y u_Y)^2 + P_4(Y_0 d_Y^2)^2\} + P_3\{P_1(Y_0 u_Y^2)^2 + P_4(Y_0 u_Y d_Y)^2\} \\
&+ P_4\{(P_1 + P_3)(Y_0 d_Y u_Y)^2 + (P_2 + P_4)(Y_0 d_Y^2)^2\} \\
&- E[Y_T 1_{S_T^s \geq L_s}]^2, \\
Var[S_T^s Y_T 1_{S_T^s < L_s}] =& (S_0^s d_s^2)^2 Y_0^2 (P_2 d_Y^2 + P_3 u_Y^2)^2 - E[S_T^s Y_T 1_{S_T^s < L_s}]^2, \\
Var[V_t 1_{V_t \geq 0, S_T^s \geq L_s}] =& (P_1 + P_3(P_1 + P_4))(V_t^u)^2 - E[V_t 1_{V_t \geq 0, S_T^s \geq L_s}]^2, \\
Var[V_t 1_{V_t < 0, S_T^l \geq L_l}] =& (P_2 + P_4(P_1 + P_2))(V_t^d)^2 - E[V_t 1_{V_t < 0, S_T^l \geq L_l}]^2, \\
Cov[Y_T 1_{S_T^s \geq L_s}, V_t 1_{V_t \geq 0, S_T^s \geq L_s}] =& V_t^u P_1\{(P_1 + P_3) Y_0 u_Y^2 + (P_3 + P_4) Y_0 u_Y d_Y\} \\
&+ V_t^u P_3 (P_1 Y_0 u_Y^2 + P_4 Y_0 u_Y d_Y) - E[Y_T 1_{S_T^s \geq L_s}]E[V_t 1_{V_t \geq 0, S_T^s \geq L_s}], \\
Cov[Y_T 1_{S_T^s \geq L_s}, V_t 1_{V_t < 0, S_T^l \geq L_l}] =& \{P_1(P_2 + P_4) Y_0 d_Y u_Y + 2P_2 P_4 Y_0 d_Y^2\} V_t^d - E[Y_T 1_{S_T^s \geq L_s}]E[V_t 1_{V_t < 0, S_T^l \geq L_l}], \\
Cov[S_T^s Y_T 1_{S_T^s < L_s}, V_t 1_{V_t < 0, S_T^l \geq L_l}] =& E[S_T^s Y_T 1_{S_T^s < L_s} V_t 1_{V_t < 0, S_T^l \geq L_l}] - E[S_T^s Y_T 1_{S_T^s < L_s}]E[V_t 1_{V_t < 0, S_T^l \geq L_l}] \\
=& P_2(P_2 + P_3) S_0^s d_s^2 V_t^d - E[S_T^s Y_T 1_{S_T^s < L_s}]E[V_t 1_{V_t < 0, S_T^l \geq L_l}],
\end{aligned}$$

$$
\begin{aligned}
E[g_Y(T)g_\perp(T)] =& P_1\left\{(P_1+P_3)Y_0u_Y^2+(P_2+P_4)Y_0u_Yd_Y-\frac{B_T^c}{B_t^c}\phi V_t^u\right\}\\
&+P_2\left\{\left(P_1+P_3\eta_s\frac{S_0^sd_s^2}{L_s}\right)Y_0d_Yu_Y+\left(P_2\eta_s\frac{S_0^sd_s^2}{L_s}+P_4\right)Y_0d_Y^2-\frac{B_T^c}{B_t^c}\phi V_t^d\right\}\\
&+P_3\Bigg\{P_1\left(Y_0u_Y^2-\frac{B_T^c}{B_t^c}\phi V_t^u\right)+P_2Y_0u_Yd_Y\eta_s\frac{S_0^sd_s^2}{L_s}+P_3Y_0u_Y^2\eta_s\frac{S_0^sd_s^2}{L_s}\eta_l\frac{S_0^ld_l^2}{L_l}\\
&+P_4\left(Y_0u_Yd_Y-\frac{B_T^c}{B_t^c}\phi V_t^u\right)\eta_l\frac{S_0^ld_l^2}{L_l}\Bigg\}\\
&+P_4\Bigg\{P_1\left(Y_0d_Yu_Y-\frac{B_T^c}{B_t^c}\phi V_t^d\right)+P_2\left(Y_0d_Y^2-\frac{B_T^c}{B_t^c}\phi V_t^d\right)\\
&+P_3Y_0d_Yu_Y\eta_l\frac{S_0^ld_l^2}{L_l}+P_4Y_0d_Y^2\eta_l\frac{S_0^ld_l^2}{L_l}\Bigg\}.
\end{aligned}
$$

And, the covariance between S_T^j and $g_{swp}(T)$ is

$$
Cov[S_T^j, g_{swp}(T)] = Cov[S_T^j, g_Y(T)] - K\cdot Cov[S_T^j, g_\perp(T)]
$$

where

$$
\begin{aligned}
Cov[S_T^j, g_Y(T)] =& Cov[S_T^j, Y_T1_{S_T^s\geq L_s}]+\frac{\eta_s}{L_s}Cov[S_T^j, S_T^sY_T1_{S_T^s<L_s}]\\
&-\frac{B_T^c}{B_t^c}\phi Cov[S_T^j, V_t1_{V_t\geq 0, S_T^s\geq L_s}]-\frac{B_T^c}{B_t^c}\phi Cov[S_T^j, V_t1_{V_t<0, S_T^l\geq L_l}],\\
Cov[S_T^j, g_\perp(T)] =& E[S_T^jg_\perp(T)]-E[S_T^j]E[g_\perp(T)],
\end{aligned}
$$

with

$$
\begin{aligned}
Cov[S_T^l, Y_T1_{S_T^s\geq L_s}] =& E[S_T^lY_T1_{S_T^s\geq L_s}]-E[S_T^l]E[Y_T1_{S_T^s\geq L_s}]\\
=& P_1(P_1S_0^lu_l^2Y_0u_Y^2+P_2S_0^lu_l^2Y_0u_Yd_Y+P_3S_0^lu_ld_lY_0u_Y^2+P_4S_0^lu_ld_lY_0u_Yd_Y)\\
&+P_2(P_1S_0^lu_l^2Y_0d_Yu_Y+P_4S_0^lu_ld_lY_0d_Y^2)+P_3(P_1S_0^ld_lu_lY_0u_Y^2+P_4S_0^ld_l^2Y_0u_Yd_Y)\\
&+P_4(P_1S_0^ld_lu_lY_0d_Yu_Y+P_2S_0^ld_lu_lY_0d_Y^2+P_3S_0^ld_l^2Y_0d_Yu_Y+P_4S_0^ld_l^2Y_0d_Y^2)\\
&-E[S_T^l]E[Y_T1_{S_T^s\geq L_s}],\\
Cov[S_T^s, Y_T1_{S_T^s\geq L_s}] =& E[S_T^sY_T1_{S_T^s\geq L_s}]-E[S_T^s]E[Y_T1_{S_T^s\geq L_s}]\\
=& P_1(P_1S_0^su_s^2Y_0u_Y^2+P_2S_0^su_sd_sY_0u_Yd_Y+P_3S_0^su_sd_sY_0u_Y^2+P_4S_0^su_s^2Y_0u_Yd_Y)\\
&+P_2(P_1S_0^sd_su_sY_0d_Yu_Y+P_4S_0^sd_su_sY_0d_Y^2)+P_3(P_1S_0^sd_su_sY_0u_Y^2+P_4S_0^sd_su_sY_0u_Yd_Y)\\
&+P_4(P_1S_0^su_s^2Y_0d_Yu_Y+P_2S_0^su_sd_sY_0d_Y^2+P_3S_0^su_sd_sY_0d_Yu_Y+P_4S_0^su_s^2Y_0d_Y^2)\\
&-E[S_T^s]E[Y_T1_{S_T^s\geq L_s}],\\
E[S_T^l] =& S_0^l\{(P_1+P_2)u_l+(P_3+P_4)d_l\}^2,\\
E[S_T^s] =& S_0^s\{(P_1+P_4)u_s+(P_2+P_3)d_s\}^2,
\end{aligned}
$$

$$
\begin{aligned}
Cov[S_T^l, S_T^sY_T1_{S_T^s<L_s}] =& E[S_T^lS_T^sY_T1_{S_T^s<L_s}]-E[S_T^l]E[S_T^sY_T1_{S_T^s<L_s}],\\
=& S_0^lS_0^sd_s^2Y_0(P_2u_ld_Y+P_3d_lu_Y)^2-E[S_T^l]E[S_T^sY_T1_{S_T^s<L_s}],\\
Cov[S_T^s, S_T^sY_T1_{S_T^s<L_s}] =& E[S_T^sS_T^sY_T1_{S_T^s<L_s}]-E[S_T^s]E[S_T^sY_T1_{S_T^s<L_s}],\\
=& (S_0^sd_s^2)^2Y_0(P_2d_Y+P_3u_Y)^2-E[S_T^s]E[S_T^sY_T1_{S_T^s<L_s}],
\end{aligned}
$$

$$
\begin{aligned}
Cov[S_T^l, V_t 1_{V_t\geq 0, S_T^s\geq L_s}] =& E[S_T^l V_t 1_{V_t\geq 0, S_T^s\geq L_s}] - E[S_T^l]E[V_t 1_{V_t\geq 0, S_T^s\geq L_s}],\\
=& P_1\{(P_1+P_2)S_0^l u_l^2 V_t^u + (P_3+P_4)S_0^l u_l d_l V_t^u\}\\
&+ P_3(P_1 S_0^l d_l u_l V_t^u + P_4 S_0^l d_l^2 V_t^u) - E[S_T^l]E[V_t 1_{V_t\geq 0, S_T^s\geq L_s}],\\
Cov[S_T^s, V_t 1_{V_t\geq 0, S_T^s\geq L_s}] =& E[S_T^s V_t 1_{V_t\geq 0, S_T^s\geq L_s}] - E[S_T^s]E[V_t 1_{V_t\geq 0, S_T^s\geq L_s}],\\
=& P_1\{(P_1+P_4)S_0^s u_s^2 V_t^u + (P_2+P_3)S_0^s u_s d_s V_t^u\}\\
&+ P_3(P_1+P_4)S_0^s d_s u_s V_t^u - E[S_T^s]E[V_t 1_{V_t\geq 0, S_T^s\geq L_s}],\\
Cov[S_T^l, V_t 1_{V_t< 0, S_T^l\geq L_l}] =& E[S_T^l V_t 1_{V_t< 0, S_T^l\geq L_l}] - E[S_T^l]E[V_t 1_{V_t< 0, S_T^l\geq L_l}],\\
=& P_2\{(P_1+P_2)S_0^l u_l^2 V_t^d + (P_3+P_4)S_0^l u_l d_l V_t^d\}\\
&+ P_4(P_1+P_2)S_0^l d_l u_l V_t^d - E[S_T^l]E[V_t 1_{V_t< 0, S_T^l\geq L_l}],\\
Cov[S_T^s, V_t 1_{V_t< 0, S_T^l\geq L_l}] =& E[S_T^s V_t 1_{V_t< 0, S_T^l\geq L_l}] - E[S_T^s]E[V_t 1_{V_t< 0, S_T^l\geq L_l}],\\
=& P_2\{(P_1+P_4)S_0^s d_s u_s V_t^d + (P_2+P_3)S_0^s d_s^2 V_t^d\}\\
&+ P_4(P_1 S_0^s u_s^2 V_t^d + P_2 S_0^s u_s d_s V_t^d) - E[S_T^s]E[V_t 1_{V_t< 0, S_T^l\geq L_l}],
\end{aligned}
$$

$$
\begin{aligned}
E[S_T^l g_\perp(T)] =& (P_1+P_2)\{(P_1+P_2)S_0^l u_l^2 + (P_3+P_4)S_0^l u_l d_l\}\\
&+ (P_3+P_4)\left\{(P_1+P_2)S_0^l d_l u_l + (P_3+P_4)S_0^l d_l^2 \eta_l \frac{S_0^l d_l^2}{L_l}\right\},\\
E[S_T^s g_\perp(T)] =& P_1\{(P_1+P_4)S_0^s u_s^2 + (P_2+P_3)S_0^s u_s d_s\} + P_2\{(P_1+P_4)S_0^s d_s u_s + (P_2+P_3)S_0^s d_s^2\}\\
&+ P_3\left\{P_1 S_0^s d_s u_s + P_2 S_0^s d_s^2 + P_3 S_0^s d_s^2 \eta_l \frac{S_0^l d_l^2}{L_l} + P_4 S_0^s d_s u_s \eta_l \frac{S_0^l d_l^2}{L_l}\right\}\\
&+ P_4\left\{P_1 S_0^s u_s^2 + P_2 S_0^s u_s d_s + P_3 S_0^s u_s d_s \eta_l \frac{S_0^l d_l^2}{L_l} + P_4 S_0^s u_s^2 \eta_l \frac{S_0^l d_l^2}{L_l}\right\}.
\end{aligned}
$$

Finally, the variance of $\frac{B_T}{B_t}\phi V_t + g_{swp}(T)$ is

$$
Var\left[\frac{B_T}{B_t}\phi V_t + g_{swp}(T)\right] = \left(\frac{B_T}{B_t}\phi\right)^2 Var[V_t] + Var\left[g_{swp}(T)\right] + 2\left(\frac{B_T}{B_t}\phi\right) Cov[V_t, g_{swp}(T)],
$$

with

$$
\begin{aligned}
E[V_t] &= (P_1+P_3)V_t^u + (P_2+P_4)V_t^d,\\
Var[V_t] =& E[V_t^2] - E[V_t]^2 = (P_1+P_3)(V_t^u)^2 + (P_2+P_4)(V_t^d)^2 - E[V_t]^2,
\end{aligned}
$$

$$
\begin{aligned}
Cov[V_t, g_{swp}(T)] =& E[V_t g_{swp}(T)] - E[V_t]E[g_{swp}(T)] \\
=& E[V_t g_Y(T)] - KE[V_t g_\perp(T)] - E[V_t]E[g_{swp}(T)] \\
=: & \text{(I)} - K \times \text{(II)} - E[V_t]E[g_{swp}(T)], \\
\text{(I)} =& E\left[V_t Y_T \left\{1 - \left(1 - \eta_s \frac{S_T^s}{L_s}\right) 1_{S_T^s < L_s}\right\}\right] - \frac{B_T^c}{B_t^c}\phi E[V_t^2(1_{V_t \geq 0, S_T^s \geq L_s} + 1_{V_t < 0, S_T^l \geq L_l})] \\
=: & \text{(III)} - \frac{B_T^c}{B_t^c}\phi \times \text{(IV)}, \\
\text{(III)} =& P_1 V_t^u \{(P_1 + P_3) Y_0 u_Y^2 + (P_2 + P_4) Y_0 u_Y d_Y\} \\
& + P_2 V_t^d \left\{P_1 Y_0 d_Y u_Y + P_2 Y_0 d_Y^2 \eta_s \frac{S_0^s d_s^2}{L_s} + P_3 Y_0 d_Y u_Y \eta_s \frac{S_0^s d_s^2}{L_s} + P_4 Y_0 d_Y^2\right\} \\
& + P_3 V_t^u \left\{\left(P_1 + P_3 \eta_s \frac{S_0^s d_s^2}{L_s}\right) Y_0 u_Y^2 + \left(P_2 \eta_s \frac{S_0^s d_s^2}{L_s} + P_4\right) Y_0 u_Y d_Y\right\} \\
& + P_4 V_t^d \{(P_1 + P_3) Y_0 d_Y u_Y + (P_2 + P_4) Y_0 d_Y^2\}, \\
\text{(IV)} =& E[V_t^2(1_{V_t \geq 0, S_T^s \geq L_s} + 1_{V_t < 0, S_T^l \geq L_l})] \\
=& (P_1 + P_3(P_1 + P_4))(V_t^u)^2 + (P_2 + P_4(P_1 + P_2))(V_t^d)^2, \\
\text{(II)} =& E\left[V_t \left\{1 - \left(1 - \eta_l \frac{S_T^l}{L_l}\right) 1_{S_T^l < L_l}\right\}\right] \\
=& P_1 V_t^u + P_2 V_t^d + (P_3 V_t^u + P_4 V_t^d)\left\{P_1 + P_2 + (P_3 + P_4)\eta_l \frac{S_0^l d_l^2}{L_l}\right\},
\end{aligned}
$$

$$
\begin{aligned}
Cov[S_T^l, V_t] =& E[S_T^l V_t] - E[S_T^l]E[V_t] \\
=& (P_1 V_t^u + P_2 V_t^d)\{(P_1 + P_2) S_0^l u_l^2 + (P_3 + P_4) S_0^l u_l d_l\} \\
& + (P_3 V_t^u + P_4 V_t^d)\{(P_1 + P_2) S_0^l d_l u_l + (P_3 + P_4) S_0^l d_l^2\} - E[S_T^l]E[V_t], \\
Cov[S_T^s, V_t] =& E[S_T^s V_t] - E[S_T^s]E[V_t] \\
=& (P_1 V_t^u + P_4 V_t^d)\{(P_1 + P_4) S_0^s u_s^2 + (P_2 + P_3) S_0^s u_s d_s\} \\
& + (P_2 V_t^d + P_3 V_t^u)\{(P_1 + P_4) S_0^s d_s u_s + (P_2 + P_3) S_0^s d_s^2\} - E[S_T^s]E[V_t].
\end{aligned}
$$