An Equilibrium Asset Pricing Model under Ambiguity

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Abstract

We consider an equilibrium in a pure exchange market under the smooth ambiguity model introduced by Klibanoff et al. (2005). We derive the state price density in the equilibrium and the capital asset pricing model by applying the dual theory for the smooth ambiguity model (Iwaki and Osaki (2014)).

Key words: Ambiguity, Capital Asset Pricing Model, Equilibrium, Smooth ambiguity model

1 Introduction

We consider an equilibrium in a pure exchange market under the smooth ambiguity model introduced by Klibanoff et al. (2005). We derive the state price density in the equilibrium and the capital asset pricing model (CAPM) by applying the dual theory for the smooth ambiguity model (Iwaki and Osaki (2014)). Historically, the first attempts at developing equilibrium models for the CAPM under risk go back to Sharpe (1964) and Lintner (1965). Although Maccheroni et al. (2013) also derived the CAPM under the smooth ambiguity model, they derived it by applying the classical mean-variance approach under risk to the smooth ambiguity model using quadratic approximation, whereas we derive it without using approximations.

In our model, individuals can achieve equilibrium allocations by trading securities in a complete market. This role of securities in spanning risk in the complete market was already recognized by Arrow (1952). Radner (1972) established the existence of equilibria in a discrete-time dynamic market with several agents who trade with one another. Lucas (1978) set up a discrete-time Markov model in which the optimal consumption/production of a representative agent leads to equilibrium.

The approach to the questions of the existence of equilibria is adapted from Karatzas et al. (1990, 1991) and Karatzas and Shreve (1998). The fundamental idea of assigning weights to the different agents, and thereby reducing the problem to one of finding the proper weights, was firstly used by Negishi (1960).

2 The model

We consider a single-period pure exchange economy. All of the uncertainty is described by a finite discrete state space $\Omega = \{\omega_1, \cdots, \omega_n\}$. To hedge uncertainty, agents in the economy trade a risk-less security whose rate of return is constant r as well as n risky securities. We assume that rate of return of the i-th security is given by a random variable $\tilde{x}_i$ such that

$$\begin{aligned} \tilde{x}_i &= \mu_i + \tilde{z}_i, \\ \tilde{z}_i &= \begin{cases} z_{i1} & \text{if } \omega_1 \text{ occurs,} \\ & \vdots \\ z_{in} & \text{if } \omega_n \text{ occurs,} \quad i = 1, \cdots, n, \end{cases} \end{aligned}$$

where μ_i is a constant. That is, we assume that the security market is complete. The ambiguity is represented by a finite set of probability distributions on Ω such that

$$\left\{ \boldsymbol{p}^{(\ell)} = (p_1^{(\ell)}, \cdots, p_n^{(\ell)});\ \ell \in \{1, \cdots, \mathcal{L}\} \right\}.$$

We assume a unique probability distribution $\boldsymbol{\rho} = (\rho^{(1)}, \cdots, \rho^{(\mathcal{L})})$ on the index $\ell \in \{1, \cdots, \mathcal{L}\}$ is given.

We define a probability distribution P on Ω by

$$\begin{aligned} P &= (P_1, \cdots, P_n) \\ &= \left(\sum_{\ell=1}^{\mathcal{L}} p_1^{(\ell)} \rho^{(\ell)}, \cdots, \sum_{\ell=1}^{\mathcal{L}} p_n^{(\ell)} \rho^{(\ell)} \right). \end{aligned}$$

We also assume that $\mathbb{E}^P[\tilde{z}_i] = \sum_{s=1}^n P_s z_{is} = 0$, $i = 1, \cdots, n$, without loss of generality.

We define the state price density as a non-negative random variable $H = (H_1, \cdots, H_n)$ on Ω satisfying

$$\mathbb{E}^P[(1+r)H] = (1+r) \sum_{s=1}^n P_s H_s = 1, \tag{1}$$

$$\mathbb{E}^P[(1+\tilde{x}_i)H] = \sum_{s=1}^n P_s (1 + x_{is}) H_s = 1, \quad i = 1 \cdots, n, \tag{2}$$

where $x_{is} = \mu_i + z_{is}$.

There exist K agents in the economy. Let ϵ_{ks}, $k = 1, \cdots, K$, denote agent k's income paid at termination if the state ω_s, $s = 1, \cdots, n$, occurs, let $w_k \geq 0$ denote agent k ' s initial wealth, and let π_{ki} denote the amount of money that agent k has invested in the i-th risky security. For each state $\omega_s \in \Omega$, the terminal wealth W_{ks} of agent k is assumed to be given by

$$\begin{aligned} W_k = \begin{pmatrix} W_{k1} \\ \vdots \\ W_{kn} \end{pmatrix} &= (1+r) w_k \mathbf{1}_n + (\boldsymbol{X} - r \mathbf{1}_{n\times n})\, \boldsymbol{\pi}_k + \boldsymbol{\epsilon}_k \\ &= (1+r) w_k \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} + \sum_{i=1}^n \pi_{ki} \begin{pmatrix} x_{i1} - r \\ \vdots \\ x_{in} - r \end{pmatrix} + \begin{pmatrix} \epsilon_{k1} \\ \vdots \\ \epsilon_{kn} \end{pmatrix}, \end{aligned} \tag{3}$$

where $\boldsymbol{\pi}_k = (\pi_{k1}, \cdots, \pi_{kn})^\top$, $\boldsymbol{\epsilon}_k = (\epsilon_{k1}, \cdots, \epsilon_{kn})^\top$,

$$\boldsymbol{X} = \begin{pmatrix} x_{11} & \cdots & x_{n1} \\ \vdots & \ddots & \vdots \\ x_{1n} & \cdots & x_{nn} \end{pmatrix},$$

and $\mathbf{1}_n = (1, \cdots, 1)^\top$ and $\mathbf{1}_{n\times n}$ are respectively an n-dimensional vector and an $n \times n$-matrix, both consisting entirely of 1s. The portfolio $\boldsymbol{\pi}_k$ for each agent k is called *admissible* for his initial wealth w_k if

$$W_k \geq 0. \tag{4}$$

We assume that the assumptions of the dual theory of the smooth ambiguity model (Iwaki and Osaki (2014)) hold and that agent k evaluates his terminal wealth by

$$\mathbb{E}^{Q_k}[u_k(W_k)] = \sum_{s=1}^{n} Q_{ks} u_k(W_{ks}), \tag{5}$$

where $Q_k = (Q_{k1}, \cdots, Q_{kn})$ is a probability distribution on Ω defined by

$$Q_{ks} = \sum_{\ell=1}^{\mathcal{L}} p_s^{(\ell)} \rho_k^{(\ell)},$$

with a probability distribution $(\rho_k^{(1)}, \cdots, \rho_k^{(\mathcal{L})})$ on the index of the first-order probability distributions $\boldsymbol{p}^{(\ell)}$, $\ell \in \{1, \cdots, \mathcal{L}\}$ that is given by agent k's attitude towards ambiguity (see Corollary 1 of Iwaki and Osaki (2014)). In (5), $u_k : (0, \infty) \to \mathbb{R}$ is a utility function that is strictly increasing, strictly concave and continuously differentiable, with $u_k'(\infty) = \lim_{x\to\infty} \frac{\mathrm{d}}{\mathrm{d}x} u_k(x) = 0$ and $u_k'(0+) = \lim_{x\downarrow 0} \frac{\mathrm{d}}{\mathrm{d}x} u_k(x) = \infty$.

For a given initial wealth w_k, agent k chooses an admissible portfolio so as to maximize his welfare represented by (5) over the class of portfolios

$$\mathcal{A}(w_k) = \left\{ \boldsymbol{\pi}_k : \ \mathbb{E}^P[H(W_k - \epsilon_k)] \leq w_k, \ \mathbb{E}^{Q_k}[u_k^-(W_k)] < \infty \right\}. \tag{6}$$ [1]

In other words, each agent computes the value function

$$V_k(w_k) = \sup_{\pi_k \in \mathcal{A}(w_k)} \mathbb{E}_k^Q[u_k(W_k)], \tag{7}$$

and seeks to find an optimal portfolio $\hat{\pi}_k \in \mathcal{A}(w_k)$ that attains the corresponding supremum.

To solve the problem, we define

$$\mathcal{X}_k(y) = \mathbb{E}^{Q_k}\left[HL_k^{-1}\left(I_k(yL_k^{-1}H) - \boldsymbol{\epsilon}_k\right)\right], \tag{8}$$

where L_k is the likelihood ratio defined by $L_k = (L_{k1}, \cdots, L_{kn}) = \left(\frac{Q_{k1}}{P_1}, \cdots, \frac{Q_{kn}}{P_n}\right)$, and I_k is the inverse function of the marginal utility u_k'. We note that I_k is a map from $(0, \infty)$ onto itself with $I_k(0+) = u_k'(0+) = \infty$ and $I_k(\infty) = u_k'(\infty) = 0$. We also note that the convex dual $\tilde{u}_k$ of u_k, defined by

$$\tilde{u}_k(y) = \max_{x \in (0,\infty)}[u_k(x) - xy], \quad y \in (0, \infty), \tag{9}$$

is a decreasing, convex and continuously differentiable function on $(0, \infty)$ and satisfies

$$\tilde{u}_k(y) = u_k(I_k(y)) - yI_k(y), \tag{10}$$

$$\tilde{u}_k'(y) = -I_k(y), \quad y \in (0, \infty). \tag{11}$$

Proposition 1. *Suppose that*

$$\mathcal{X}_k(y) < \infty, \quad y \in (0, \infty) \tag{12}$$

and that

$$V_k(x) < \infty, \quad x \in (0, \infty).$$

[1] $u_k^-(W_k)$ denotes the negative part of the portfolio, that is, $u_k^-(W_k) = \max\{0, -u_k(W_k)\}$.

Then agent k's optimal wealth $\hat{W}_k$ and optimal portfolio $\hat{\pi}_k$ are given by

$$\hat{W}_{ks} = I_k\left(y_k L_{ks}^{-1} H_s\right), \ s = 1, \cdots, n, \tag{13}$$

$$\hat{\boldsymbol{\pi}}_k = (\boldsymbol{X} - r\mathbf{1}_{n\times n})^{-1} \begin{pmatrix} \hat{W}_{k1} - (1+r)w_k - \epsilon_{k1} \\ \vdots \\ \hat{W}_{kn} - (1+r)w_k - \epsilon_{kn} \end{pmatrix}, \tag{14}$$

where y_k is a solution to the equation

$$\begin{aligned} &\mathcal{X}_k(y_k) = w_k \\ \Longleftrightarrow &\sum_{s=1}^{n} P_s H_s \left(I_k(y_k L_{ks}^{-1} H_s) - \epsilon_{ks}\right) = w_k, \end{aligned} \tag{15}$$

and $(\boldsymbol{X} - r\mathbf{1}_{n\times n})^{-1}$ is the inverse matrix of $(\boldsymbol{X} - r\mathbf{1}_{n\times n})$.[2]

Proof. Because $\hat{W}_{ks} \geq 0$, $\hat{\boldsymbol{\pi}}_k$ is an admissible portfolio. We first show that $\hat{\boldsymbol{\pi}}_k \in \mathcal{A}(w_k)$. From (9) and (10), and because $\hat{W}_{ks} \geq 0$,

$$u_k(\hat{W}_k) \geq u_k(1) + y_k L_k^{-1} H(\hat{W}_k - 1) \geq u_k(1) - y_k L_k^{-1} H.$$

Hence,

$$\begin{aligned} \mathbb{E}^{Q_k}[u_k^-(\hat{W}_{ks})] \leq &|u_k(1)| + y_k \mathbb{E}^{Q_k}[L_k^{-1} H] \\ = &|u_k(1)| + y_k \mathbb{E}^{P}[H] \leq |u_k(1)| + \frac{y_k}{1+r} < \infty. \end{aligned}$$

That is, we have $\hat{\boldsymbol{\pi}}_k \in \mathcal{A}(w_k)$.

Next we show that $\hat{\boldsymbol{\pi}}_k$ is optimal. For all $y > 0$ and $\boldsymbol{\pi}_k \in \mathcal{A}(w_k)$,

$$\begin{aligned} \mathbb{E}^{Q_k}[u_k(W_k)] \leq &\mathbb{E}^{Q_k}[u_k(W_k)] + y\left\{w_k - \mathbb{E}^{Q}[L_k^{-1} H(W_k - \boldsymbol{\epsilon}_k)]\right\} \\ \leq &\mathbb{E}^{Q_k}[\tilde{u}_k(y L_k^{-1} H)] + y\left(w_k + \mathbb{E}^{Q}[L_k^{-1} H \boldsymbol{\epsilon}_k]\right) \\ = &\mathbb{E}^{Q_k}[u_k(I_k(y L_k^{-1} H))] + y\left\{w_k - \mathbb{E}^{Q}[L_k^{-1} H(I_k(y L_k^{-1} H) - \boldsymbol{\epsilon}_k)]\right\}. \end{aligned}$$

The above inequalities become equalities if and only if $y = y_k$ and $W_k = \hat{W}_k$.

[2]Because the market is complete by assumption, this inverse matrix exists.

This means that $\boldsymbol{\pi}_k = \hat{\boldsymbol{\pi}}_k$ is optimal from (14). $\square$

3 An Equilibrium

Definition 1. *An equilibrium is defined as a set of pairs* $(\hat{\boldsymbol{\pi}}_k, \hat{W}_k)$, $k = 1, \cdots, K$, *of an optimal portfolio and terminal wealth satisfying the following equations:*

$$\sum_{k=1}^{K} \hat{\boldsymbol{\pi}}_k = \mathbf{0},$$
$$\sum_{k=1}^{K} \hat{W}_k = (1+r)w_0 \mathbf{1}_n + \boldsymbol{\varepsilon}, \tag{16}$$

where w_0 *and* $\boldsymbol{\varepsilon} = (\varepsilon_1, \cdots, \varepsilon_n)^\top$ *are the aggregate initial wealth and the aggregate terminal income defined by* $w_0 = \sum_{k=1}^{K} w_k$ *and* $\varepsilon_s = \sum_{k=1}^{K} \epsilon_{ks}$, $s = 1, \cdots, n$, *respectively.*

From (13), it follows that (16) is equivalent to

$$\sum_{k=1}^{K} I_k(y_k L_{ks}^{-1} H_s) = w_0(1+r) + \varepsilon_s, \quad s = 1, \cdots, n. \tag{17}$$

For an arbitrarily given $\mathbf{\Gamma} = (\gamma_1, \cdots, \gamma_K) \in \mathbb{R}_{++}^K$ and for each $s = 1, \cdots, n$, let the function $\mathcal{I}_s(\cdot;\, \mathbf{\Gamma})$ be defined by

$$\mathcal{I}_s(y;\, \mathbf{\Gamma}) := \sum_{k=1}^{K} I_k\left(\frac{y}{\gamma_k L_{ks}}\right).$$

Then (17) is equivalent to

$$\mathcal{I}_s(H_s; \mathbf{\Gamma}) = w_0(1+r) + \varepsilon_s, \quad s = 1, \cdots, n,$$

with $\gamma_k = \frac{1}{y_k}$. Hence defining the inverse function $\mathcal{H}_s(\cdot; \mathbf{\Gamma})$ of $\mathcal{I}_s(\cdot; \mathbf{\Gamma})$ by

$$\mathcal{I}_s(\mathcal{H}_s(x; \mathbf{\Gamma}); \mathbf{\Gamma}) = x, \quad s = 1, \cdots, n, \tag{18}$$

the state price density in the equilibrium is given by

$$H_s = \mathcal{H}_s(w_0(1+r) + \varepsilon_s; \mathbf{\Gamma}),$$

and the budget constraint (15) is equivalent to

$$\sum_{s=1}^{n} P_s \mathcal{H}_s(w_0(1+r) + \epsilon_s; \mathbf{\Gamma}) \left[I_k \left(\frac{\mathcal{H}_s(w_0(1+r) + \epsilon_s; \mathbf{\Gamma})}{\gamma_k L_{ks}} \right) - \epsilon_{ks} \right] = w_k. \tag{19}$$

In other words, deriving the equilibrium is equivalent to deriving $\mathbf{\Gamma}$ to satisfy (19).

Next we consider the existence of the equilibrium.

Proposition 2. *There exists some $\mathbf{\Gamma} \in (0, \infty)^K$ satisfying (19).*

Proof. Let $\mathbb{K} = \{1, \cdots, K\}$ and let $\boldsymbol{e}_1, \cdots, \boldsymbol{e}_K \in \mathbb{R}^K$ be the K-dimensional unit vectors. For $\mathbb{B} \subset \mathbb{K}$, we define

$$S_{\mathbb{B}} = \left\{ \sum_{k \in \mathbb{B}} \gamma_k \boldsymbol{e}_k : \sum_{k \in \mathbb{B}} \gamma_k = 1,\ \gamma_k \geq 0, k \in \mathbb{B} \right\},$$

$$S_{\mathbb{B}}^{+} = \left\{ \sum_{k \in \mathbb{B}} \gamma_k \boldsymbol{e}_k : \sum_{k \in \mathbb{B}} \gamma_k = 1,\ \gamma_k > 0, k \in \mathbb{B} \right\}.$$

For each $k \in \mathbb{K}$, we define a function $R_k : S_{\mathbb{K}} \to \mathbb{R}$ by

$$R_k(\mathbf{\Gamma}) = \begin{cases} \sum_{s=1}^{n} P_s \mathcal{H}_s(w_0(1+r) + \varepsilon_s; \mathbf{\Gamma}) \left(I_k \left(\frac{\mathcal{H}_s(w_0(1+r)+\varepsilon_s; \mathbf{\Gamma})}{\gamma_k L_{ks}} \right) - \epsilon_{ks} \right) - w_k \\ \qquad \text{if } \gamma_k > 0, \\ -w_k \quad \text{if } \gamma_k = 0. \end{cases}$$

To prove that the proposition holds, we have to show that there exists $\mathbf{\Gamma} \in S_{\mathbb{K}}$ satisfying $R_k(\mathbf{\Gamma}) = 0$ for each $k \in \mathbb{K}$.

Because the function R_k is continuous, the set

$$F_k = \{\mathbf{\Gamma} \in S_{\mathbb{K}};\ R_k(\mathbf{\Gamma}) \geq 0\}$$

is closed. On the other hand, from (17) and (18),

$$\sum_{k \in \mathbb{K}} R_k(\mathbf{\Gamma}) = 0, \quad \mathbf{\Gamma} \in S_{\mathbb{K}}. \tag{20}$$

Now, suppose that there exists $\hat{\mathbf{\Gamma}} \in S_{\mathbb{K}}$ such that $\hat{\mathbf{\Gamma}} \notin \cup_{k \in \mathbb{K}} F_k$. Then $R_k(\hat{\mathbf{\Gamma}}) < 0$, which contradicts (20). This leads to

$$S_{\mathbb{K}} = \cup_{k \in \mathbb{K}} F_k.$$

Furthermore, suppose that there exists $\hat{\mathbf{\Gamma}} \in S_{\mathbb{B}}$ such that $\hat{\mathbf{\Gamma}} \notin \cup_{k \in \mathbb{B}} F_k$. Then $R_k(\hat{\mathbf{\Gamma}}) < 0$ for all $k \in \mathbb{B}$. In this case, because $R_j(\hat{\mathbf{\Gamma}}) < 0$ for all $j \in \mathbb{K}$ such that $\hat{\gamma}_j = 0$ and $\hat{\mathbf{\Gamma}} \in S_{\mathbb{K}}$,

we have $\sum_{k \in \mathbb{K}} R_k(\hat{\mathbf{\Gamma}}) < 0$ which again contradicts (20). This leads to

$$S_{\mathbb{B}} \subset \cup_{k \in \mathbb{B}} F_k \quad \forall\, \mathbb{B} \in \mathbb{K}. \tag{21}$$

From (21) and the Knaster-Kratowski-Mazurkiewicz Theorem (cf. Border (1985), p. 26), the set $\cap_{k \in \mathbb{K}} F_k$ is nonempty. For any $\mathbf{\Gamma}^* \in \cap_{k \in \mathbb{K}} F_k$, we have

$$R_k(\mathbf{\Gamma}^*) = 0 \qquad \forall\, k \in \mathbb{K}, \tag{22}$$

for otherwise we would have $\sum_{k \in \mathbb{K}} R_k(\mathbf{\Gamma}^*) > 0$, which contradicts (20), and we also have $\mathbf{\Gamma}^* \in S_{\mathbb{K}}^+$, for if there exists $k \in \mathbb{K}$ such that $\gamma_k = 0$ then we have $R_k(\mathbf{\Gamma}^*) < 0$, which contradicts (22). Therefore we can conclude that $\mathbf{\Gamma}^* \in \cap_{k \in \mathbb{K}} F_k$ belongs to $(0, \infty)^K$. □

For each $\mathbf{\Gamma}_s = (\gamma_1 L_{1s}, \cdots, \gamma_K L_{Ks})$, $s = 1, \cdots, n$, we define the utility function of the representative agent by

$$U(x;\mathbf{\Gamma}_s) = \max_{x_k} \sum_{k=1}^{K} \gamma_k L_{ks} u_k(x_k).$$

Proposition 3.

$$U'(w_0(1+r)+\varepsilon_s, \mathbf{\Gamma}_s) = \mathcal{H}_s(w_0(1+r)+\varepsilon_s; \mathbf{\Gamma}).$$

Proof. Let w_{ks}^* be defined by

$$w_{ks}^* = I_k((\gamma_k L_{ks})^{-1}\mathcal{H}_s(w_0(1+r)+\varepsilon_s;\mathbf{\Gamma})). \tag{23}$$

Then

$$\begin{aligned}\sum_{k=1}^{K} w_{ks}^* &= \sum_{k=1}^{K} I_k((\gamma_k L_{ks})^{-1}\mathcal{H}_s(w_0(1+r)+\varepsilon_s;\mathbf{\Gamma})) \\ &= \mathcal{I}_s(\mathcal{H}_s(w_0(1+r)+\varepsilon_s;\mathbf{\Gamma});\mathbf{\Gamma}) = w_0(1+r)+\varepsilon_s.\end{aligned}$$

From (23),

$$\gamma_k L_{ks} u_k'(w_{ks}^*) = \mathcal{H}_s(w_0(1+r)+\varepsilon_s;\mathbf{\Gamma}),$$

and because u_k is strictly concave,

$$\begin{aligned}\sum_{k=1}^{K} \gamma_k L_{ks} u_k(x_k) &< \sum_{k=1}^{K} \gamma_k L_{ks} \left\{u_k(w_{ks}^*) + (x_k - w_{ks}^*)u_k'(w_{ks}^*)\right\} \\ &= \sum_{k=1}^{K} \gamma_k L_{ks} u_k(w_{ks}^*) + \mathcal{H}_s(w_0(1+r)+\varepsilon_s;\mathbf{\Gamma}) \sum_{k=1}^{K}(x_k - w_{ks}^*).\end{aligned}$$

Hence,

$$U(w_0(1+r)+\varepsilon_s;\mathbf{\Gamma}_s) = \sum_{k=1}^{K} \gamma_k L_{ks} u_k \left(I_k\left(\frac{\mathcal{H}_s(w_0(1+r)+\varepsilon_s;\mathbf{\Gamma})}{\gamma_k L_{ks}}\right)\right).$$

Differentiating the above equation completes the proof.[3] □

Let κ be the index of absolute risk aversion for the representative agent, defined by

$$\kappa(x;\mathbf{\Gamma}_s) = -\frac{U''(x;\mathbf{\Gamma}_s)}{U'(x;\mathbf{\Gamma}_s)}, \qquad s = 1,\cdots,n. \tag{24}$$

We immediately obtain the following corollary.

Corollary 1.

$$\mathcal{H}_s(w_0(1+r)+\varepsilon_s;\mathbf{\Gamma}) = \frac{\exp\left(-\int_0^{w_0(1+r)+\varepsilon_s}\kappa\left(x;\mathbf{\Gamma}_s\right)dx\right)}{(1+r)\mathbb{E}^P\left[\exp\left(-\int_0^{w_0(1+r)+\varepsilon_s}\kappa\left(x;\mathbf{\Gamma}\right)dx\right)\right]}.$$

Proof. From (24), there is a constant C for which

$$U'(x;\mathbf{\Gamma}_s) = C\exp\left(-\int_0^x \kappa\left(t;\mathbf{\Gamma}_s\right)dt\right).$$

The result then follows from Proposition 3 and the fact that $\mathbb{E}^P[U'(w_0(1+r)+\varepsilon_s;\mathbf{\Gamma}] = \frac{1}{1+r}$. □

We note that Corollary 1 is an extension of the general economic premium principle of Bühlmann under risk (Bühlmann (1984)) to that under uncertainty.

Proposition 4.

$$\mu_i - r = -\frac{\mathbb{E}^P[U'(w_0(1+r)+\varepsilon_s;\mathbf{\Gamma})z_i]}{\mathbb{E}^P[U'(w_0(1+r)+\varepsilon_s;\mathbf{\Gamma})]}. \tag{25}$$

[3] We use the fact from (18) that

$$\frac{\mathrm{d}}{\mathrm{d}x}\sum_{k=1}^{K} I_k\left(\frac{\mathcal{H}_s(x;\mathbf{\Gamma})}{\gamma_k L_{ks}}\right) = 1.$$

Proof. From (1) and (2),

$$\begin{aligned}
\mathbb{E}^P[\mathcal{H}(w_0(1+r)+\varepsilon_s;\mathbf{\Gamma})] &= \frac{1}{1+r},\\
1 &= \mathbb{E}^P[\mathcal{H}(w_0(1+r)+\varepsilon_s;\mathbf{\Gamma})(1+\mu_i+\tilde{z}_i)]\\
&= \mathbb{E}^P[\mathcal{H}(w_0(1+r)+\varepsilon_s;\mathbf{\Gamma})(1+r+\mu_i-r+\tilde{z}_i)]\\
&= 1+\mathbb{E}^P[\mathcal{H}(w_0(1+r)+\varepsilon_s;\mathbf{\Gamma})](\mu_i-r)+\mathbb{E}^P[\mathcal{H}(w_0(1+r)+\varepsilon_s;\mathbf{\Gamma})\tilde{z}_i].
\end{aligned}$$

Hence,

$$\mu_i - r = -\frac{\mathbb{E}^P[\mathcal{H}(w_0(1+r)+\varepsilon_s;\mathbf{\Gamma})\tilde{z}_i]}{\mathbb{E}^P[\mathcal{H}(w_0(1+r)+\varepsilon_s;\mathbf{\Gamma})]}.$$

□

Let $\boldsymbol{\pi}_M = (\pi_{M1}, \cdots, \pi_{Mn})^\top$ be a portfolio satisfying

$$\begin{pmatrix} U'(w_0(1+r)+\varepsilon_s,\mathbf{\Gamma}_1) \\ \vdots \\ U'(w_0(1+r)+\varepsilon_s,\mathbf{\Gamma}_n) \end{pmatrix} = (\boldsymbol{X} - r\mathbf{1}_{n\times n})\boldsymbol{\pi}_M. \tag{26}$$

We note that there exists a unique $\boldsymbol{\pi}_M$ because the market is complete. We refer to $\boldsymbol{\pi}_M$ as *the market portfolio*.

Corollary 2. *Let μ_M be the expected rate of return of the market portfolio, that is, $\mu_M = \frac{\sum_{i=1}^n \pi_{Mi}\mu_i}{\sum_{i=1}^n \pi_{Mi}}$. Let $R_i = \mu_i + \tilde{z}_i$ be the rate of return of the i-th risky security and let $R_M = \frac{\sum_{i=1}^n \pi_{Mi}R_i}{\sum_{i=1}^n \pi_{Mi}}$ be the rate of return of the market portfolio.*

Then

$$\mu_i - r = \beta_i(\mu_M - r),$$

where

$$\beta_i = \frac{\mathrm{Cov}(R_i, R_M)}{\mathrm{Var}(R_M)},$$

and Cov *and* Var *denote the covariance and the variance under P, respectively.*

Proof. From (25) and (26),

$$\begin{aligned}\mu_i - r &= -\frac{\mathbb{E}^P[U'(w_0(1+r)+\varepsilon_s;\mathbf{\Gamma})z_i]}{\mathbb{E}^P[U'(w_0(1+r)+\varepsilon_s;\mathbf{\Gamma})]}\\ &= -\frac{\mathbb{E}^P[\sum_{j=1}^n z_j\pi_{Mj}z_i]}{\mathbb{E}^P[U'(w_0(1+r)+\varepsilon_s;\mathbf{\Gamma})]},\\ \mu_M - r &= \sum_{i=1}^n \frac{\pi_{Mi}}{\sum_{j=1}^n \pi_{Mj}}(\mu_i - r)\\ &= -\frac{\mathbb{E}^P[(\sum_{j=1}^n \pi_{Mj}z_j)^2]}{\sum_{j=1}^n \pi_{Mj}\mathbb{E}^P[U'(w_0(1+r)+\varepsilon_s;\mathbf{\Gamma})]}.\end{aligned}$$

Hence

$$\mu_i - r = \sum_{j=1}^n \pi_{Mj}\frac{\mathbb{E}^P[\sum_{j=1}^n z_j\pi_{Mj}z_i]}{\mathbb{E}^P[(\sum_{j=1}^n \pi_{Mj}z_j)^2]}(\mu_M - r).$$

On the other hand, because $\mathrm{Cov}(R_i, R_M) = \frac{1}{\sum_{j=1}^n \pi_{Mj}}\mathbb{E}^P[z_i\sum_{j=1}^n \pi_{Mj}z_j]$ and $\mathrm{Var}(R_M) = \frac{1}{(\sum_{j=1}^n \pi_{Mj})^2}\mathbb{E}^P[(\sum_{j=1}^n \pi_{Mj}z_j)^2]$,

$$\beta_i = \sum_{j=1}^n \pi_{Mj}\frac{\mathbb{E}^P[z_i\sum_{j=1}^n \pi_{Mj}z_j]}{\mathbb{E}^P[(\sum_{j=1}^n \pi_{Mj}z_j)^2]}.$$

Hence we obtain the result. □

We note that Corollary 2 says the CAPM (Sharpe (1964) and Lintner (1965)) holds under uncertainty in the model.

4 Two-fund Separation

We first show that each agent's optimal portfolio can be decomposed as a sum of the market portfolio and his specific portfolio. From (14) and (26), the following proposition holds.

Proposition 5. *For agent k, the optimal portfolio $\hat{\boldsymbol{\pi}}_k$ can be decomposed as a sum* *of the market portfolio $\boldsymbol{\pi}_M$ and his specific portfolio $\boldsymbol{\pi}_k^e$ as follows:*

$$\hat{\boldsymbol{\pi}}_k = \boldsymbol{\pi}_M + \boldsymbol{\pi}_k^e, \tag{27}$$

where

$$\boldsymbol{\pi}_k^e = (\boldsymbol{X} - r\mathbf{1}_{n\times n})^{-1} \begin{pmatrix} \hat{W}_{k1} - (1+r)w_k - \epsilon_{k1} - U'(w_0(1+r)+\varepsilon_1; \Gamma_1) \\ \vdots \\ \hat{W}_{kn} - (1+r)w_k - \epsilon_{kn} - U'(w_0(1+r)+\varepsilon_n; \Gamma_n) \end{pmatrix}.$$

We note that if all of agents have quadratic utility functions, are ambiguity neutral and their terminal incomes are proportional to the aggregate income, then $\boldsymbol{\pi}_k^e$ disappears from the decomposition (27).

Corollary 3. *Assume that all agents have quadratic utility functions, that they are all ambiguity neutral, and that all their terminal incomes are proportional to the aggregate income ε. Then the optimal portfolio for each agent consists of the market portfolio and the risk-less security.*

Proof. Because all of the agents have a quadratic utility function, we can assume that the marginal utility for each agent k, $k = 1, \cdots, K$, is given by

$$u_k'(x) = -x + \alpha_k, \quad x < \alpha_k.$$

Noting that ambiguity neutrality implies that $L_{ks} = 1$, $s = 1, \cdots, n$, it follows from (13) that

$$\hat{W}_{ks} = \alpha_k - y_k H_s. \tag{28}$$

Because the terminal income is proportional to the aggregate income ε, there exists a constant δ_k satisfying $\epsilon_k = \delta_k \varepsilon$ and $\sum_{k=1}^K \delta_k = 1$. Hence, from (15),

$$y_k = \frac{\frac{\alpha_k}{1+r} - w_k - \delta_k \mathbb{E}^P[H\varepsilon]}{\mathbb{E}^P[H^2]}. \tag{29}$$

Substituting (29) into (28), we have from (3) that

$$
\begin{aligned}
\hat{W}_{ks} &= \alpha_k - \frac{\frac{\alpha_k}{1+r} - w_k - \delta_k \mathbb{E}^P[H\varepsilon]}{\mathbb{E}^P[H^2]} H_s \\
&= (1+r)w_k + (x_{1s} - r, \cdots, x_{ns} - r)\boldsymbol{\pi}_k + \delta_k \varepsilon_s \\
\iff & \left((1+r) - \frac{H_s}{\mathbb{E}^P[H^2]}\right)\left(\frac{\alpha_k}{1+r} - w_k\right) + \delta_k \left(\frac{\mathbb{E}^P[H\varepsilon]H_s}{\mathbb{E}^P[H^2]} - \varepsilon_s\right) \\
&= (x_{1s} - r, \cdots, x_{ns} - r)\boldsymbol{\pi}_k. \qquad (30)
\end{aligned}
$$

Summing the above equation for $k = 1, \cdots, K$ and applying (16), we have

$$
\frac{\mathbb{E}^P[H\varepsilon]H_s}{\mathbb{E}^P[H^2]} - \varepsilon_s = -\left((1+r) - \frac{H_s}{\mathbb{E}^P[H^2]}\right)\zeta,
$$

where $\zeta = \sum_{k=1}^K \left(\frac{\alpha_k}{1+r} - w_k\right)$. Substituting this into (30), we have

$$
\left((1+r) - \frac{H_s}{\mathbb{E}^P[H^2]}\right)\left(\frac{\alpha_k}{1+r} - w_k - \zeta\delta_k\right) = (x_{1s} - r, \cdots, x_{ns} - r)\boldsymbol{\pi}_k.
$$

Hence, from Proposition 3 and by the definition of the market portfolio $\boldsymbol{\pi}_M$, the optimal portfolio $\hat{\boldsymbol{\pi}}_k$ satisfies

$$
(x_{1s} - r, \cdots, x_{ns} - r)\hat{\boldsymbol{\pi}}_k = \iota_k \left((1+r) - \frac{1}{\mathbb{E}^P[H^2]}(x_{1s} - r, \cdots, x_{ns} - r)\boldsymbol{\pi}_M\right),
$$

where $\iota_k = \frac{\alpha_k}{1+r} - w_k - \zeta\delta_k$. That is, the optimal portfolio consists of the risk-less security and the market portfolio.

□

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