Variance premium, U-shaped Pricing Kernel and Option Valuation

Abstract

This study extends the works of Christoffersen et al. (2013) and Christoffersen et al. (2008) to investigate the pricing kernel of index options incorporating variance term structure. The variance term structure is incorporated not only in the physical processes but also in the variance-dependent pricing kernel to account for the variance premiums. The physical process generalizes the model Heston and Nandi (2000), Duan (1995), and Christoffersen et al. (2008). The obtained pricing kernels are U-shaped and more flexible than the one of Christoffersen et al. (2013). We use the in-sample and out-sample test to verify the predictability of this generalized model.

Keywords: pricing kernel; variance premium; volatility term structure; option valuation.

1. Introduction

After the classical Black-Scholes model originally developed, financial literatures contribute significant improvements on index option valuation. However, many pricing puzzles still exist between empirical studies and theoretical models. Option researchers try to reconcile the pricing gap between observed market price and theoretical value in two fields: more comprehensive physical dynamics and more flexible pricing kernels.

When modeling the stochastic process, we often use physical measure to model the spot price, and risk-neutral measure to model the option prices. The connection between physical and risk-neutral measure in existing theory is often called as pricing kernel, stochastic discount factor or state price density, was developed by Rubinstein (1976) and Brennan (1979). In literatures, those pricing kernels were set as a monotonic function of market return. However, some problem followed arising. First, as Brown and Jackwerth (2001) pointed out that pricing kernel was not always monotonic function of return. So the monotonic setting was seems not satisfy anymore. Second, a problem came from Bates (1996), to find balance of distributions between physical measure and risk-neutral measure is difficult.

For solving the these problems, Christoffersen et al. (2013) provided a more general pricing kernel than the Rubinstein (1976) and Brennan (1979), and extended the GARCH model to satisfy the general pricing kernel. The GARCH option valuation model was developed by Heston and Nandi (2000). It is a valuation formula for a spot asset whose variance follows a GARCH process. The empirical tests show that new pricing kernel can solve several problems above, and gives a better accuracy when fitting the option prices.

Furthermore, another reason causing option bias is considered as the shortcoming of modeling the term structure of volatility. Christoffersen et al. (2008) present a multiple volatility components model which derives from GARCH model in Heston and Nandi (2000). The model allows long-run volatility as time-varying to description richer volatility dynamics. Compared to GARCH model, the multiple volatility components model has superior performance and ability to model the volatility term structure.

This paper tries to combine the advantages of the multiple volatility components model and the new pricing kernel. We first establish the pricing kernel which has two important properties as like the pricing kernel in Christoffersen et al. (2013). One is the volatility premium also consisting of equity risk premium and volatility preferences. Another is the natural logarithm of adjusting pricing kernel is still a quadratic function of the market return. Based on the two literatures, we provide two different analyses of the model. First we derive the risk-neutral measure of the model by pricing kernel and then estimate the parameters by maximum likelihood estimation (MLE). Our comparison uses the RMSE and bias between two models. The empirical result is quite striking. Both RMSE and bias are better than before. The second part, we split data into in-sample and out-sample for testing the predictability of model. When using model with pricing kernel estimation, the RMSEs of both in- and out-of-sample period are all less than the original.

This paper organizes as follows. Section 2 presents the adjusted pricing kernel and the multi volatility component model in both physical and risk-neutral measure. Section 3 shows the estimation of new multi volatility component model. Section 4 is the results of empirical tests. And Section 5 concludes.

2. Model

2.1 Multiple Volatility Components Model

Christoffersen et al. (2008) propose a multiple volatility components model to enhance the GARCH option valuation model of Heston and Nandi (2000) and Christoffersen et al. (2008). The model consists of multi volatility which is given by

$$\ln[S_t] = \ln[S_{t-1}] + r + \left(\mu - \frac{1}{2}\right) h_t + \sqrt{h_t} z_t \tag{1a}$$

where S_t is the price of underlying asset, r denotes daily continuously compounded interest rate, the parameter μ governs the equity premium, and z_t is assumed to be standard normal distribution. h_t is the total conditional volatility which consists of two components: $h_t - q_t$ the short-run conditional variance and q_t the long-run conditional variance:

$$h_t = q_t + \tilde{\beta}(h_{t-1} - q_{t-1}) + \alpha[\left(z_{t-1} - \gamma_1\sqrt{h_{t-1}}\right)^2 - (1 + \gamma_1^2 h_{t-1})] \tag{1b}$$

$$q_t = \omega + \rho q_{t-1} + \varphi[\left(z_{t-1} - \gamma_2\sqrt{h_{t-1}}\right)^2 - (1 + \gamma_2^2 h_{t-1})], \tag{1c}$$

where $\tilde{\beta}$ and ρ are the coefficients of the autoregressive terms. α and φ are the level of influence caused by stochastic term. γ_1 and γ_2 parameters resulted in asymmetric influence of shocks. According to the literature, several properties as following, first, the expected future variance is

$$E_{t-1}[h_{t+1}] = \omega + \rho q_t + \tilde{\beta}(h_t - q_t). \tag{2a}$$

It can be dispersed as a part of long-run and short-run expected future volatility

$$E_{t-1}[h_{t+1} - q_{t+1}] = \tilde{\beta}(h_t - q_t) \tag{2b}$$

$$E_{t-1}[q_{t+1}] = \omega + \rho q_t. \tag{2c}$$

Second, The conditional variance of the h_{t+1} is

$$\mathrm{Var}_{t-1}[h_{t+1}] = 2(\alpha + \varphi)^2 + 4(\alpha\gamma_1 + \varphi\gamma_2)^2 h_t. \tag{3a}$$

It also can be dispersed as part of long-run and short-run conditional variance of volatility

$$\mathrm{Var}_{t-1}[h_{t+1} - q_{t+1}] = 2\alpha^2 + 4\alpha^2{\gamma_1}^2 h_t. \tag{3b}$$

$$\mathrm{Var}_{t-1}[q_{t+1}] = 2\varphi^2 + 4\varphi^2{\gamma_2}^2 h_t. \tag{3c}$$

Third, the conditional correlation between the volatility (h_{t+1}) and stock return (R_t), is

$$\mathrm{Corr}_{t-1}(R_t, h_{t+1}) = \frac{-2(\alpha\gamma_1+\varphi\gamma_2)\sqrt{h_t}}{\sqrt{2(\alpha+\varphi)^2+4(\alpha\gamma_1+\varphi\gamma_2)^2 h_t}}. \tag{4a}$$

Similarly, the conditional correlation of short-run and long volatility is

$$\mathrm{Corr}_{t-1}(R_t, h_{t+1} - q_{t+1}) = \frac{-2\gamma_1\sqrt{h_t}}{\sqrt{2+4{\gamma_1}^2 h_t}}; \tag{4b}$$

$$\mathrm{Corr}_{t-1}(R_t, q_{t+1}) = \frac{-2\gamma_2\sqrt{h_t}}{\sqrt{2+4{\gamma_2}^2 h_t}}. \tag{4c}$$

We construct the form of pricing kernel similar to Christoffersen et al. (2013), namely

$$\frac{M_t}{M_0} = \left[\frac{S_t}{S_0}\right]^{\phi} \exp\{\delta t + \eta_1 \textstyle\sum_{s=1}^{t}(h_s - q_s) + \eta_2 \sum_{s=1}^{t} q_s + \xi_1[(h_{t+1} - q_{t+1}) - (h_1 - q_1)] + \xi_2[q_{t+1} - q_1]\}, \tag{5}$$

to satisfy the multiple volatility components model in (1a)–(1c). Comparing to literature, our pricing kernel decomposes the volatility into long-run and short-run term and gives new parameters. Therefore, a risk-neutral process can be given as the Proposition 1.

Proposition 1 The risk-neutral stock price process corresponding to the physical multiple volatility component process (1a)–(1c) and the pricing kernel (5) is also a multiple volatility component process.

$$\ln[S_t] = \ln[S_{t-1}] + r - \frac{1}{2}h_t^* + \sqrt{h_t^*}z_t^* \tag{6a}$$

$$h_t^* = q_t^* + \tilde{\beta}^*(h_{t-1}^* - q_{t-1}^*) + \alpha^*\left\{\left[z_{t-1}^* - \gamma_1^*\sqrt{h_{t-1}^*}\right]^2 - \left[1 - 2(\alpha\xi_1 + \varphi\xi_2) + {\gamma_1^*}^2 h_{t-1}^*\right]\right\} \tag{6b}$$

$$q_t^* = \omega^* + \rho^* q_{t-1}^* + \varphi^* \left\{ \left[z_{t-1}^* - \gamma_2^* \sqrt{h_{t-1}^*} \right]^2 - \left[1 - 2(\alpha\xi_1 + \varphi\xi_2) + \gamma_2^{*2} h_{t-1}^* \right] \right\}, \tag{6c}$$

where z_t^* is a standard normal distribution and the parameter denoted with * in (6a)–(6c) have a relationship with same parameter in (1a) –(1c), namely

$$h_t^* = \frac{h_t}{1-2(\alpha\xi_1+\varphi\xi_2)} \tag{7a}$$

$$q_t^* = \frac{q_t}{1-2(\alpha\xi_1+\varphi\xi_2)} \tag{7b}$$

$$\omega^* = \frac{\omega}{1-2(\alpha\xi_1+\varphi\xi_2)} \tag{7c}$$

$$\alpha^* = \frac{\alpha}{[1-2(\alpha\xi_1+\varphi\xi_2)]^2} \tag{7d}$$

$$\varphi^* = \frac{\varphi}{[1-2(\alpha\xi_1+\varphi\xi_2)]^2} \tag{7e}$$

$$\gamma_1^* = \left(\mu + \gamma_1 - \frac{1}{2}\right) \cdot [1 - 2(\alpha\xi_1 + \varphi\xi_2)] + \frac{1}{2} \tag{7f}$$

$$\gamma_2^* = \left(\mu + \gamma_2 - \frac{1}{2}\right) \cdot [1 - 2(\alpha\xi_1 + \varphi\xi_2)] + \frac{1}{2} \tag{7g}$$

$$\tilde{\beta}^* = \tilde{\beta} + (\alpha^* \gamma_1^{*2} - \alpha\gamma_1^2) + (\varphi^* \gamma_2^{*2} - \varphi\gamma_2^2) \tag{7h}$$

$$\rho^* = \rho + (\alpha^* \gamma_1^{*2} - \alpha\gamma_1^2) + (\varphi^* \gamma_2^{*2} - \varphi\gamma_2^2), \tag{7i}$$

Proof. See Appendix A. ■

Comparing to equation (1) and equations (6a)–(6c), the risk-neutral dynamics has a lot of change due to the influence of both equity premium parameter μ and scaling factor $1 - 2(\alpha\xi_1 + \varphi\xi_2)$. But the form of two dynamics is quite similar. If ξ_1 and ξ_2 are zeros, the dynamic will back to the risk-neutral process in Christoffersen et al. (2008). Furthermore, a little different with Christoffersen et al. (2013) but interesting result is the scaling factor $1 - 2(\alpha\xi_1 + \varphi\xi_2)$ playing direct role in the model. It can be thought as different measurement units between two dynamics. And the multiplier relationship between dynamics is also the scaling factor.

2.2 Implications of the Variance-Dependent Pricing Kernel

We can observe the new pricing kernel causing a great change compared to original risk-neutral dynamics. Not only parameter but also level, persistence and variance of volatility differs in both measures with the nonzero $(\alpha\xi_1 + \varphi\xi_2)$. When assuming $(\alpha\xi_1 + \varphi\xi_2) > 0$, by equations (7a) and (7b), both risk-neutral long-run and short-run volatility exceeds the physical long-run and short-run volatility. Other impacts are displayed in the expectation future volatility, conditional variance of volatility and conditional correlation between return and volatility, as following

$$E_{t-1}[h_{t+1}^*] = \omega^* + \rho^* q_t^* + \tilde{\beta}^* (h_t^* - q_t^*) + 2(\alpha\xi_1 + \varphi\xi_2)(\alpha^* + \varphi^*), \tag{8}$$

$$\mathrm{Var}_{t-1}[h_{t+1}^*] = 2(\alpha^* + \varphi^*)^2 + 4\left(\alpha^*\gamma_1^* + \varphi^*\gamma_2^*\right)^2 h_t^*, \text{ and} \tag{9}$$

$$\mathrm{Corr}_{t-1}[R_t, h_{t+1}^*] = \frac{-2\left(\alpha^*\gamma_1^* + \varphi^*\gamma_2^*\right)\sqrt{h_t^*}}{\sqrt{2(\alpha^* + \varphi^*)^2 + 4\left(\alpha^*\gamma_1^* + \varphi^*\gamma_2^*\right)^2 h_t^*}}. \tag{10}$$

Certainly the expectation, variance and correlation can be dispersed to a part of long-run or short-run volatility. The detail of the properties of long-run or short-run volatility is written in Appendix.

Here we summarize the impact between physical and risk-neutral measures in Corollary 1.

Corollary 1 If

(i). The correlations between return and volatility are negative $(\gamma_1, \gamma_2 > 0)$,

(ii). Equity premium is positive $(\mu > 0)$,

(iii). $1 - 2\left(\alpha\xi_1 + \varphi\xi_2\right) < 1$.

Then,

(i). The risk-neutral variance exceeds the physical variance,

(ii). The risk-neutral expected future volatility exceeds the physical measure expected future volatility, (iii). The persistence in risk-neutral is higher than in physical measure,

(iv). The risk-neutral variance of volatility exceeds the physical variance of volatility.

Notice that the corollary not only applies to the total volatility, but also applies to the component of long-run or short-run volatility. Next, we check out whether the adjusted pricing kernel is still a quadratic function of stock return or not.

Corollary 2 The logarithm of the pricing kernel is a quadratic function of the stock return.

$$\ln\left[\frac{M_t}{M_{t-1}}\right] = \frac{(\alpha\xi_1+\varphi\xi_2)}{h_t}(R_t - r)^2 - \mu(R_t - r) + \frac{1}{2}\left[\left(\mu - \frac{1}{2}\right)^2 - \frac{1}{4[1-2(\alpha\xi_1+\varphi\xi_2)]}\right]h_t - r + \frac{1}{2}\ln[1 - 2(\alpha\xi_1 + \varphi\xi_2)].$$

Proof. See Appendix C. ▪

By taking the similar form to Christoffersen et al. (2013), we confirm that the adjusted pricing kernel still remain the properties of quadratic function of the stock return.

Corollary 3 When scaling factor $1 - 2(\alpha\xi_1 + \varphi\xi_2)$ is less than one, the pricing kernel has a U-shape.

Considering if $1 - 2(\alpha\xi_1 + \varphi\xi_2) < 1$, then $\alpha\xi_1 + \varphi\xi_2$ will be positive. So quadratic term in Corollary 2 will also be positive which means the opening direction of the parabola is upward. So the adjusted pricing kernel still remains the U-shape properties when under the condition.

2.3 Option Valuation

The ultimate purpose of this paper is the valuation of derivatives on the underlying asset by considering the new pricing kernel with volatility components model. We use the same way as Christoffersen et al. (2008). First, we derive the moment generating function and then extend to the risk-neutral dynamics. The option valuation techniques use the result of Heston and Nandi (2000) by characterizing the MGF of the log stock price. The details are as following.

The MGF is defined by

$$g^*_{t,T}(\theta) \equiv E^*_t\{\exp[\theta \ln(S_T)]\}$$

$$= \exp\left[\theta \ln(S_t) + A_t + B_{1,t}(h^*_{t+1} - q^*_{t+1}) + B_{2,t} q^*_{t+1}\right], \quad (11)$$

Where the coefficients that

$$A_t = A_{t+1} + r\theta + \omega^* B_{2,t+1} - \frac{1}{2}\ln\left(1 - 2\alpha^* B_{1,t+1} - 2\varphi^* B_{2,t+1}\right) - [1 - 2(\alpha\xi_1 + \varphi\xi_2)][\alpha^* B_{1,t+1} + \varphi^* B_{2,t+1}];$$

$$B_{1,t} = \tilde{\beta}^* B_{1,t+1} - \frac{1}{2}\theta + 2\frac{[\alpha^*\gamma_1^* B_{1,t+1} + \varphi^*\gamma_2^* B_{2,t+1} - 0.5\theta]^2}{1 - 2\alpha^* B_{1,t+1} - 2\varphi^* B_{2,t+1}};$$

$$B_{2,t} = \rho^* B_{2,t+1} - \frac{1}{2}\theta + 2\frac{[\alpha^*\gamma_1^* B_{1,t+1} + \varphi^*\gamma_2^* B_{2,t+1} - 0.5\theta]^2}{1 - 2\alpha^* B_{1,t+1} - 2\varphi^* B_{2,t+1}},$$

and the terminal conditions is $A_T = B_{1,T} = B_{2,T} = 0$. By the MGF and risk-neutral dynamics, option valuation formula can be presented as

$$C(S_t, h^*_{t+1}, X, T) = S_t P_{1,t} - X\exp[-r(T - t)]P_{2,t},$$

where C is the function of call price, S_t is the stock price in t time, X is the strike price of the contract, T is the maturity of the contract, r is the risk free rate using the term structure of interest rates, and

$$P_{1,t} = \frac{1}{2} + \frac{1}{\pi}\exp[-r(T - t)] \int_0^\infty \mathrm{Re}\left[\frac{X^{-i\theta} g^*_{t,T}(i\theta + 1)}{i\theta S_t}\right] d\theta,$$

$$P_{2,t} = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{X^{-i\theta} g^*_{t,T}(i\theta)}{i\theta}\right] d\theta.$$

Function $g^*_{t,T}(i\theta)$ in above is the conditional characterize function of the logarithm of the spot price under the risk-neutral and $\mathrm{Re}[]$ represents the real part of complex number. For calculating the numerical integral, we treat the integration interval of $[0, \infty)$ by using the 32 points Gaussian-Laguerre Quadrature.

3. Empirical Analysis

3.1 Data

The empirical analysis uses the S&P500 index options data adapted from OptionMetrics. The sample period starts from January 1, 1996 to August 31, 2013. Following Bakshi, Cho and Chen (1997), we use several standard procedures to filter the data:

I. Because of the long cross-section of options data, we follow Dumas, Fleming and Whaley (1998) and Heston and Nandi (2000) using Wednesday data to avoid the day-of-the-week effects. Another reason is that the Wednesday data are considered the least likely to be a holiday (Christoffersen et al., 2013).

II. Only use the contracts with positive trading volume and the maturities are between two weeks to one year.

III. For each maturity, we keep six contracts of strike price of largest trading volume.

IV. We convert put price into call by put-call-parity and only keep the option contract which the price is large than $0.375.

V. Remove the data that not satisfy the no-arbitrage condition

$$C_{t,\tau} \geq \max[0, S_t - K, S_t - \overline{D}_{t,\tau} - K \cdot B_{t,\tau}],$$

where τ is the maturity, and $\overline{D}_{t,\tau} = \sum_{s=1}^{\tau-t} D_{t+s} \cdot \exp[-R_{t,s} \cdot s]$ is the total present value of dividend.

VI. Lastly, we only keep out-of-money options by moneyness. Moneyness is defined as implied futures price (F) divided by strike price (X). The implied futures price can be acquired from Black model. If F/X smaller than one, the contract is the out-money call. If F/X larger than one, the contract is the out-money put.

Table 1 displays the descriptive statistics of stock index and option contract data.

The S&P 500 sample return includes 4,448 days with an average of 8.67% and standard deviation of 20.21%. The return have a small negative skewness and moderate excess kurtosis. The total numbers of option contracts is 30,043 with an average implied volatility of 20.42% and average price (defined as $0.5 \cdot (\text{ask} + \text{bid})$) of $23.41. The volatility smirk phenomenon can also be observed in Table 1.

Table 1.

Descriptive statistics of stock index and option data

S&P500 index

Number of Return	Mean	Standard Deviation	Skewness	Kurtosis
4448	8.67 %	20.21%	-0.042	7.517

Option Data Sorted by Moneyness

Moneyness	Number of Contract	Average IV	Average Price	Average Spread
$F/X < 0.96$	10947	20.24%	10.5144	1.4552
$0.96 < F/X \leq 0.98$	3445	17.65%	17.4924	1.4703
$0.98 < F/X \leq 1.02$	6256	17.55%	27.8333	1.6620
$1.02 < F/X \leq 1.04$	1634	18.46%	38.1827	1.9098
$1.04 < F/X \leq 1.06$	1373	19.54%	34.5565	1.7576
$1.06 < F/X$	6388	25.88%	20.1037	1.5292
Total	30043	20.42%	23.4128	1.5985

Option Data Sorted by Days to Maturity

Maturity	Number of Contract	Average IV	Average Price	Average Spread
$DTM \leq 30$	3299	19.50%	7.9340	0.7701
$30 < DTM \leq 60$	6882	19.93%	13.3151	1.1888
$60 < DTM \leq 90$	5082	20.29%	19.2176	1.5217
$90 < DTM \leq 120$	3396	21.19%	23.5178	1.7658
$120 < DTM \leq 180$	3345	20.53%	27.3487	1.8245
$180 < DTM$	8039	20.94%	39.3794	2.1731
Total	30043	20.42%	23.4128	1.5985

Notes :

The sample period starts from January 1, 1996 to August 31, 2013. The data is daily, but we only use every Wednesday and out-of-money options data. If F/X smaller than one, the contract is an out-of-money call. If F/X larger than one, the contract is an out-of-money put.

3.2 Estimation

In the estimation, we first use Maximum Likelihood method to estimate the parameters of the model which presents in Section 2. Following Christoffersen et al. (2013), the joint likelihood is composed of option data and return data. The log likelihood of return-based part is defined as

$$\ln(L_{Return}) \propto -\frac{1}{2}\sum_{t=1}^{T}\left\{\ln[h_t] + \frac{[R_t - r - (\mu - \frac{1}{2})h_t)]^2}{h_t}\right\}, \quad (12)$$

where $R_t = \ln[S_t/S_{t-1}]$ following the assumption of conditional density of daily return is a normal distribution. The log likelihood of option-based part is

$$\ln(L_{Option}) \propto -\frac{1}{2}\sum_{i=1}^{N}\left\{\ln[s_\varepsilon^2] + \frac{\varepsilon_i^2}{s_\varepsilon^2}\right\}, \quad (13)$$

where ε_i is the Black-Scholes Vega (BS Vega) weighted option valuation errors and s_ε is the population standard deviation. The Black-Scholes Vega is weighted valuation error and defined as

$$\varepsilon_i = (\text{CallPrice}_i^{Market} - \text{CallPrice}_i^{Model})/\text{BS Vega}_i^{Market},$$

where $\text{CallPrice}_i^{Market}$ is the market price of the option, $\text{CallPrice}_i^{Model}$ is the model price of the option, and $\text{BS Vega}_i^{Market}$ is the Black-Scholes Vega of the option. Assume the errors s_ε following an independent and identically normal distribution with standard deviation, we can get the log likelihood of option-based part. Because the population standard deviation (s_ε) is unknown, we use the sample one $\hat{s}_\varepsilon^2 = \frac{1}{N}\sum_{i=1}^{N}\varepsilon_i^2$ to replace it.

To estimate the variance-dependent pricing kernel, we solve the optimization problem as a combination of equation (12) and (13),

$$\max_{\psi,\ \text{scaling factor},\ \psi^*}\ [\ln L_{Return} + \ln L_{Option}], \quad (14)$$

where $\psi = \{\omega, \alpha, \tilde{\beta}, \gamma_1, \gamma_2, \rho, \varphi, \mu\}$ is the physical measure parameters and $\psi^* = \{\omega^*, \alpha^*, \tilde{\beta}^*, \gamma_1^*, \gamma_2^*, \rho^*, \varphi^*\}$ is the risk-neutral measure parameters. Note that we do not estimate the parameters ξ_1 and ξ_2 in pricing kernel. Instead, we estimate the scaling factor $1-2(\alpha\xi_1+\varphi\xi_2)$. We use this method to estimate two models, one is the original model in Christoffersen et al. (2008), and the other is the advanced model with pricing kernel in Section 2.

Table 2.
Results of estimated parameters

Parameter	Physical Parameters	
	Original Model	Advanced Model
ω	1.554E-07	4.700E-07
α	2.329E-06	7.979E-07
φ	2.753E-06	1.584E-06
$\tilde{\beta}$	0.8736	0.8979
ρ	0.9932	0.9917
γ_1	337.50	754.81
γ_2	53.56	152.00
μ	1.3924	2.1641
Parameter	**Risk Neutral Parameters**	
	Original Model	Advanced Model
$1-2(\alpha\xi_1+\varphi\xi_2)$	1.0000	0.8658
ω^*	1.554E-07	5.429E-07
α^*	2.329E-06	1.065E-06
φ^*	2.753E-06	2.114E-06
$\tilde{\beta}^*$	0.8762	0.9017
ρ^*	0.9958	0.9955
γ_1^*	338.90	655.42
γ_2^*	54.96	133.53
Total Likelihood	84424.1	85008.6
From returns	18026.6	18033.3
From options	66397.5	66975.3

Notes :
The original model is the model in Christoffersen et al. (2008) without volatility premium. The advanced model is the model with pricing kernel. We estimate the parameters by optimizing the joint likelihood consists of returns and options. In addition to these parameters, we also show the likelihood value of the two parts and sum of them.

Table 2 presents the estimation results. Compare to model in Christoffersen et al.

(2008) as benchmark and consider the likelihood for both models. The contribution from return and options is quite similar. When jointing the new pricing kernel, the likelihood from both return and option has a significant improvement. So pricing kernel considering the variance premium seems useful in the multiple volatility components model. Table 3 prints the physical properties by using equations (2), (3), and (4) and risk-neutral properties by using equations (8), (9), and (10). The parameters of model are from Table 2. According to results, we can check the conclusion of Section 2 is true.

Table 3.
Properties of Model Under Physical and Risk-Neutral Measure

Conditional Moments	Physical Distribution	
	Original Model	Advanced Model
Persistence	0.9991	0.9991
Average annual volatility	0.1751	0.1692
Variance of volatility	2.167E-05	1.821E-05
Variance of short-run volatility	1.753E-05	1.283E-05
Variance of long-run volatility	5.059E-06	5.581E-06
Correlation of return and volatility	-0.9074	-0.9700
Correlation of return and short-run volatility	-0.9676	-0.9930
Correlation of return and long-run volatility	-0.5779	-0.8737
Properties	**Risk Neutral Distribution**	
	Original Model	Advanced Model
Persistence	0.9995	0.9995
Average annual volatility	0.1753	0.1822
Variance of volatility	2.182E-05	2.280E-05
Variance of short-run volatility	1.760E-05	1.599E-05
Variance of long-run volatility	5.113E-06	7.099E-06
Correlation of return and volatility	-0.9085	-0.9661
Correlation of return and short-run volatility	-0.9678	-0.9919
Correlation of return and long-run volatility	-0.5869	-0.8624

Notes :

Here is the properties in both models, the parameters to compute is from Table 2. The original model is the model in Christoffersen et al. (2008) without volatility premium. The advanced model is the model with pricing kernel. The persistence is computed by $\rho + \tilde{\beta}(1 - \rho)$.

Table 4 presents the accuracy of model price fitting in Table 2 by using the Bias and RMSE as basis of assessment. Bias is defined as the absolute value of difference between fitting price and real price.

$$\text{Bias} = \sqrt{\frac{\sum |C_{i,\text{Mod}} - C_{i,\text{Market}}|}{N}},$$

RMSE is defined as the

$$\text{RMSE} = \sqrt{\frac{\sum (C_{i,\text{Mod}} - C_{i,\text{Market}})^2}{N}},$$

Table 4.
Option Pricing Bias and RMSE by Moneyness and Maturity.

(a) By Moneyness	Bias			RMSE		
	CJO Model	Our Model	(Normalize)	CJO Model	Our Model	(Normalize)
$F/X < 0.96$	1.992	1.970	(0.989)	2.747	2.604	(0.948)
$0.96 < F/X \leq 0.98$	3.298	3.146	(0.954)	4.599	4.252	(0.925)
$0.98 < F/X \leq 1.02$	4.651	4.327	(0.930)	6.348	5.790	(0.912)
$1.02 < F/X \leq 1.04$	5.534	5.070	(0.916)	7.299	6.637	(0.909)
$1.04 < F/X \leq 1.06$	6.898	6.537	(0.948)	9.242	8.706	(0.942)
$1.06 < F/X$	9.833	9.083	(0.924)	13.267	12.318	(0.928)
Total	5.617	5.256	(0.936)	8.533	7.924	(0.929)
(b) By Maturity	**CJO Model**	**Our Model**	**(Normalize)**	**CJO Model**	**Our Model**	**(Normalize)**
$DTM \leq 30$	5.711	5.429	(0.951)	8.340	8.021	(0.962)
$30 < DTM \leq 60$	4.770	4.368	(0.916)	7.643	6.984	(0.914)
$60 < DTM \leq 90$	5.427	4.821	(0.888)	8.393	7.368	(0.878)
$90 < DTM \leq 120$	5.780	5.174	(0.895)	9.141	8.067	(0.883)
$120 < DTM \leq 180$	6.147	5.660	(0.921)	9.905	8.824	(0.891)
$180 < DTM$	6.731	6.487	(0.964)	9.509	8.974	(0.944)
Total	5.617	5.256	(0.936)	8.533	7.924	(0.929)

Notes :
Moneyness is defined as implied future price divided by strike price. The original model is the model in Christoffersen et al. (2008) without volatility premium. The advanced model is the model with pricing kernel. The parameter is from Table 2. RMSE refers to the square root of the mean-squared valuation errors. Normalize defines as the bias/RMSE of equity and volatility premium dividing the bias/RMSE of equity premium only.

where the $C_{i,Mod}$ is the price computing by model, $C_{i,Market}$ is the price of option data and N is the numbers of sample. The fitting price of model with pricing kernel has more accuracy than original model.

Following Table 5 presents the accuracy of implied volatility which computed by parameters given in Table 2. We use the IVBias and IVRMSE as basis of assessment.

$$\mathrm{IVBias} = \sqrt{\frac{1}{N}\sum\left|\sigma(C_{i,Mod}(h_t(\psi^*))) - \sigma_{i,Market}\right|}\ ,$$

and

$$\mathrm{IVRMSE} = \sqrt{\frac{1}{N}\sum(\sigma(C_{i,Mod}(h_t(\psi^*))) - \sigma_{i,Market})^2}\ ,$$

The implied volatility of model with pricing kernel still has better performance than the original CJO model.

Table 5.
Comparison of Estimated Implied Volatility by Moneyness and Maturity

	IVBias			IVRMSE		
(a) By Moneyness	CJO Model	Our Model	Our/CJO	CJO Model	Our Model	Our/CJO
$F/X < 0.96$	3.562%	3.137%	(0.881)	4.958%	4.561%	(0.920)
$0.96 < F/X \le 0.98$	3.643%	3.259%	(0.895)	5.025%	4.640%	(0.923)
$0.98 < F/X \le 1.02$	3.850%	3.393%	(0.881)	5.053%	4.617%	(0.914)
$1.02 < F/X \le 1.04$	3.898%	3.431%	(0.880)	5.121%	4.724%	(0.923)
$1.04 < F/X \le 1.06$	4.106%	3.714%	(0.904)	5.271%	4.935%	(0.936)
$1.06 < F/X$	4.538%	4.052%	(0.893)	5.693%	5.242%	(0.921)
Total	3.988%	3.550%	(0.890)	5.246%	4.838%	(0.922)
(b) By Maturity	CJO Model	Our Model	Our/CJO	CJO Model	Our Model	Our/CJO
$DTM \le 30$	3.291%	3.195%	(0.971)	4.349%	4.302%	(0.989)
$30 < DTM \le 60$	3.174%	2.849%	(0.897)	4.128%	3.821%	(0.926)
$60 < DTM \le 90$	3.680%	3.076%	(0.836)	4.583%	4.055%	(0.885)
$90 < DTM \le 120$	4.150%	3.417%	(0.823)	5.069%	4.399%	(0.868)
$120 < DTM \le 180$	4.637%	3.851%	(0.831)	5.669%	4.964%	(0.876)
$180 < DTM$	5.815%	5.004%	(0.861)	7.420%	6.684%	(0.901)

Total	3.988%	3.550%	(0.890)	5.246%	4.838%	(0.922)

Notes :

Moneyness is defined as implied future price divided by strike price. The original model is the model in Christoffersen et al. (2008) without volatility premium. The advanced model is the model with pricing kernel. The parameter is from Table 2. RMSE refers to the square root of the mean-squared valuation errors. Normalize defines as the bias/RMSE of equity and volatility premium dividing the bias/RMSE of equity premium only.

Table 6 presents two sets of RMSEs. For instance, we use option data from January, 2002 to December, 2004 and return data from January, 1996 to December, 2004 as in-sample period for estimation. And then extend the model to out-sample period, from January, 2005 to December, 2005. We both compute the RMSE of in-sample and out-sample period. In the in-sample period, the RMSE of the model with pricing kernel is at least 99.18% of that of the benchmark original model. For the out-of-sample period, the ratio of the RMSEs (Normalized) is at least 98.14%. Although we can observe that the degree of improvement is quite different for each data set. But the generalized model more or less has a better predictability than original model.

Table 6.

The Comparison of One-Years Out-Sample Test

In-Sample Period and Out-Sample Year	RMSE		
	CJO Model	Our Model	Our/CJO
2002 - 2004	4.972	4.962	(0.998)
2005	6.358	5.770	(0.908)
2003 - 2005	5.313	5.301	(0.998)
2006	5.840	5.779	(0.990)
2004 - 2006	6.390	5.926	(0.927)
2007	13.654	11.213	(0.821)
2005 - 2007	8.444	7.850	(0.930)
2008	11.910	8.275	(0.695)
2006 - 2008	8.992	8.802	(0.979)
2009	4.897	4.819	(0.984)

2007 - 2009	6.624	6.524	(0.985)
2010	5.204	5.178	(0.995)
2008 - 2010	5.953	5.604	(0.941)
2011	9.154	9.083	(0.992)
2009 - 2011	6.171	5.921	(0.959)
2012	6.548	6.490	(0.991)

Notes: We use three years option data and all return data as in-sample period, then extend to next year as out-sample. The original model is the model in Christoffersen et al. (2008) without volatility premium. The advanced model is the model with pricing kernel. Normalized are the percentage which with respect to RMSE from original model. RMSE refers to the square root of the mean-squared valuation errors.

4. Conclusion

This paper generalizes the multiple volatility components model in Christoffersen et al. (2008) by jointing the pricing kernel containing variance premium. The empirical test, we provide two different analyses. First, we benchmark the model's performance to the original model and find that the fit of the model with the pricing kernel is better. Second, we also follow Christoffersen et al. (2008) to do an in-sample and out-sample test. The generalized model sometimes does a great improvement in predictability.

The future in this paper can be extended and generalized in a number of ways. First, the stock model can be changed as another distribution. For instance, the inverse Gaussian shock model in Christoffersen et al. (2006) could be a possible approach. Second, is any possible to find the component model similar to the relationship between GARCH model and GARCH-Jump model? If it is possible, then it also can be extended by the variance dependent pricing kernel.

Appendix

A. Proof of Proposition 1.

Rewriting equation (1) in Section 2 as

$$\ln[S_t] = \ln[S_{t-1}] + r + \left(\mu - \frac{1}{2}\right) h_t + \sqrt{h_t} z_t \quad \text{(A.1a)}$$

$$h_t = q_t + \tilde{\beta}(h_{t-1} - q_{t-1}) + \alpha[\left(z_{t-1} - \gamma_1\sqrt{h_{t-1}}\right)^2 - (1 + \gamma_1^2 h_{t-1})] \quad \text{(A.1b)}$$

$$q_t = \omega + \rho q_{t-1} + \varphi[\left(z_{t-1} - \gamma_2\sqrt{h_{t-1}}\right)^2 - (1 + \gamma_2^2 h_{t-1})], \quad \text{(A.1c)}$$

where z_t is assumed to be standard normal distribution. And therefore from equation (2), the relationship between M_t and M_{t-1} is

$$\frac{M_t}{M_{t-1}} = \left[\frac{S_t}{S_{t-1}}\right]^{\phi} \exp\{\delta + \eta_1(h_t - q_t) + \eta_2 q_t + \xi_1[(h_{t+1} - q_{t+1}) - (h_t - q_t)] + \xi_2[q_{t+1} - q_t]\}. \quad \text{(A.2)}$$

Substituting (A.1) into (A.2) gives

$$\begin{aligned}
\frac{M_t}{M_{t-1}} &= \exp\left\{\phi\left[r + \left(\mu - \frac{1}{2}\right) h_t + \sqrt{h_t} z_t\right] + \delta + \eta_1(h_t - q_t) + \eta_2 q_t\right. \\
&\quad + \xi_1\{(\tilde{\beta} - 1)(h_t - q_t) + \alpha[(z_t - \gamma_1\sqrt{h_t})^2 - (1 + \gamma_1^2 h_t)]\} \\
&\quad \left. + \xi_2\{\omega + (\rho - 1)q_t + \varphi[\left(z_t - \gamma_2\sqrt{h_t}\right)^2 - (1 + \gamma_2^2 h_t)]\}\right\} \\
&= \exp\{\phi r + \delta - \alpha\xi_1 + (\omega - \varphi)\xi_2 \\
&\quad + \left[\phi\left(\mu - \frac{1}{2}\right) + \eta_1 + \xi_1(\tilde{\beta} - 1)\right](h_t - q_t) \\
&\quad + \left[\phi\left(\mu - \frac{1}{2}\right) + \eta_2 + \xi_2(\rho - 1)\right] q_t \\
&\quad + [\phi - 2(\alpha\gamma_1\xi_1 + \varphi\gamma_2\xi_2)]\sqrt{h_t} z_t + (\alpha\xi_1 + \varphi\xi_2) z_t^2\}.
\end{aligned}$$

First, for any initial value, the parameters must be consistent with the Euler equation for the riskless asset.

$$E_{t-1}\left[\frac{M_t}{M_{t-1}}\right] = \exp(-r). \tag{A.3}$$

The conditional expectation under filter $t-1$ of $\frac{M_t}{M_{t-1}}$ is

$$\begin{aligned} E_{t-1}\left[\frac{M_t}{M_{t-1}}\right] = \exp\Big\{ & \phi r + \delta - \alpha\xi_1 + (\omega - \varphi)\xi_2 \\ & + \left[\phi\left(\mu - \frac{1}{2}\right) + \eta_1 + \xi_1(\tilde{\beta} - 1)\right](h_t - q_t) \\ & + \left[\phi\left(\mu - \frac{1}{2}\right) + \eta_2 + \xi_2(\rho - 1)\right] q_t \Big\} \\ & \cdot E_{t-1}\{[\phi - 2(\alpha\gamma_1\xi_1 + \varphi\gamma_2\xi_2)]\sqrt{h_t} z_t + (\alpha\xi_1 + \varphi\xi_2) z_t^2\}. \end{aligned} \tag{A.4}$$

We use the following formula

$$E[\exp\{az^2 + 2abz\}] = \exp\{-\frac{1}{2}\ln(1-2a) + \frac{2a^2b^2}{1-2a}\}.$$

Let

$$a = \alpha\xi_1 + \varphi\xi_2;$$

$$b = \frac{[\phi - 2(\alpha\gamma_1\xi_1 + \varphi\gamma_2\xi_2)]}{2(\alpha\xi_1 + \varphi\xi_2)}\sqrt{h_t},$$

then

$$2ab = [\phi - 2(\alpha\gamma_1\xi_1 + \varphi\gamma_2\xi_2)]\sqrt{h_t},$$

so that

$$\begin{aligned} & E_{t-1}\{[\phi - 2(\alpha\gamma_1\xi_1 + \varphi\gamma_2\xi_2)]\sqrt{h_t} z_t + (\alpha\xi_1 + \varphi\xi_2) z_t^2\} \\ & = \exp\left\{-\frac{1}{2}\ln[1 - 2(\alpha\xi_1 + \varphi\xi_2)] + \frac{[\phi - 2(\alpha\gamma_1\xi_1 + \varphi\gamma_2\xi_2)]^2 h_t}{2[1 - 2(\alpha\xi_1 + \varphi\xi_2)]}\right\}. \end{aligned}$$

Substituting the results backs into equation (A.4), so that

$$\begin{aligned} E_{t-1}\left[\frac{M_t}{M_{t-1}}\right] = \exp\Big\{ & \phi r + \delta - \alpha\xi_1 + (\omega - \varphi)\xi_2 - \frac{1}{2}\ln[1 - 2(\alpha\xi_1 + \varphi\xi_2)] + \\ & \left[\phi\left(\mu - \frac{1}{2}\right) + \eta_1 + \xi_1(\tilde{\beta} - 1) + \frac{[\phi - 2(\alpha\gamma_1\xi_1 + \varphi\gamma_2\xi_2)]^2}{2[1 - 2(\alpha\xi_1 + \varphi\xi_2)]}\right](h_t - q_t) + \\ & \left[\phi\left(\mu - \frac{1}{2}\right) + \eta_2 + \xi_2(\rho - 1) + \frac{[\phi - 2(\alpha\gamma_1\xi_1 + \varphi\gamma_2\xi_2)]^2}{2[1 - 2(\alpha\xi_1 + \varphi\xi_2)]}\right] q_t \Big\}, \end{aligned} \tag{A.5}$$

Consider equations (A.3) and (A.5), we solve δ, η_1, and η_2 by following

$$r(\phi+1)+\delta-\alpha\xi_1+(\omega-\varphi)\xi_2-\frac{1}{2}\ln[1-2(\alpha\xi_1+\varphi\xi_2)]+\Big[\phi\left(\mu-\frac{1}{2}\right)+\eta_1+$$

$$\xi_1(\tilde{\beta}-1)+\frac{[\phi-2(\alpha\xi_1\gamma_1+\varphi\xi_2\gamma_2)]^2}{2[1-2(\alpha\xi_1+\varphi\xi_2)]}\Big](h_t-q_t)+\Big[\phi\left(\mu-\frac{1}{2}\right)+\eta_2+\xi_2(\rho-1)+$$

$$\frac{[\phi-2(\alpha\xi_1\gamma_1+\varphi\xi_2\gamma_2)]^2}{2[1-2(\alpha\xi_1+\varphi\xi_2)]}\Big]q_t=0.$$

And then we can get the solution of δ, η_1, and η_2 as

$$\delta=-r(\phi+1)+\alpha\xi_1+\xi_2(\varphi-\omega)+\frac{1}{2}\ln[1-2(\alpha\xi_1+\varphi\xi_2)] \quad \text{(A.6a)}$$

$$\eta_1=-\phi\left(\mu-\frac{1}{2}\right)+\xi_1(1-\tilde{\beta})-\frac{[\phi-2(\alpha\gamma_1\xi_1+\varphi\gamma_2\xi_2)]^2}{2[1-2(\alpha\xi_1+\varphi\xi_2)]} \quad \text{(A.6b)}$$

$$\eta_2=-\phi\left(\mu-\frac{1}{2}\right)+\xi_2(1-\rho)-\frac{[\phi-2(\alpha\gamma_1\xi_1+\varphi\gamma_2\xi_2)]^2}{2[1-2(\alpha\xi_1+\varphi\xi_2)]}. \quad \text{(A.6c)}$$

Now we use the Euler equation for the underlying index

$$E_{t-1}\left[\frac{M_t}{M_{t-1}}\frac{S_t}{S_{t-1}}\right]=1, \quad \text{(A.7)}$$

to solve the parameter ϕ. In order to substitute $\frac{M_t}{M_{t-1}}$ by $\frac{M_t}{M_{t-1}}\frac{S_t}{S_{t-1}}$ for (A.7), the easy way is to replace ϕ by $\phi+1$ in equation (A.2). So the equation can written as

$$E_{t-1}\left[\frac{M_t}{M_{t-1}}\frac{S_t}{S_{t-1}}\right]=\exp\Big\{(\phi+1)r+\delta-\alpha\xi_1+\xi_2(\omega-\varphi)-\frac{1}{2}\ln[1-2(\alpha\xi_1+\varphi\xi_2)]+$$

$$\left[(\phi+1)\left(\mu-\frac{1}{2}\right)+\eta_1+\xi_1(\tilde{\beta}-1)+\frac{[1+\phi-2(\alpha\gamma_1\xi_1+\varphi\gamma_2\xi_2)]^2}{2[1-2(\alpha\xi_1+\varphi\xi_2)]}\right](h_t-q_t)+$$

$$\left[(\phi+1)\left(\mu-\frac{1}{2}\right)+\eta_2+\xi_2(\rho-1)+\frac{[1+\phi-2(\alpha\gamma_1\xi_1+\varphi\gamma_2\xi_2)]^2}{2[1-2(\alpha\xi_1+\varphi\xi_2)]}\right]q_t\Big\}.$$

(A.8)

Using the solutions of δ, η_1, and η_2 substitute into equation (A.8). We have

$$\left(\mu-\frac{1}{2}\right)+\frac{[1+2\phi-4(\alpha\gamma_1\xi_1+\varphi\gamma_2\xi_2)]}{2[1-2(\alpha\xi_1+\varphi\xi_2)]}=0.$$

Finally, we can yields

$$\phi=-\left(\mu-\frac{1}{2}\right)[1-2(\alpha\xi_1+\varphi\xi_2)]+2(\alpha\gamma_1\xi_1+\varphi\gamma_2\xi_2)-\frac{1}{2}. \quad \text{(A.9)}$$

To find the risk-neutral dynamic, using the risk-neutral density is proportional to the physical density times the pricing kernel

$$f^*_{t-1}(S_t) = \frac{f_{t-1}(S_t)\cdot M_t}{E_{t-1}[M_t]}.$$

After a lengthy and complex operation

$$f^*_{t-1}(S_t) = \frac{1}{\sqrt{2\pi}}\cdot\frac{1}{\sqrt{1-2(\alpha\xi_1+\varphi\xi_2)}}$$

$$\cdot\exp\left\{-\left\{[1-2(\alpha\xi_1+\varphi\xi_2)]z_t^2-[\phi-2(\alpha\gamma_1\xi_1+\varphi\gamma_2\xi_2)]\sqrt{h_t}z_t+\frac{[\phi-2(\alpha\gamma_1\xi_1+\varphi\gamma_2\xi_2)]^2h_t}{2[1-2(\alpha\xi_1+\varphi\xi_2)]}\right\}/2\right\}$$

$$\propto \frac{1}{\sqrt{2\pi}}\cdot\exp\left\{-\frac{1}{2}\left\{\sqrt{1-2(\alpha\xi_1+\varphi\xi_2)}\left[z_t-\frac{\phi-2(\alpha\gamma_1\xi_1+\varphi\gamma_2\xi_2)}{1-2(\alpha\xi_1+\varphi\xi_2)}\sqrt{h_t}\right]\right\}^2\right\},$$

and using equation (A.9)

$$f^*_{t-1} \propto \frac{1}{\sqrt{2\pi}}\cdot\exp\left\{-\frac{1}{2}\left\{\sqrt{1-2(\alpha\xi_1+\varphi\xi_2)}\left[z_t+(\mu+\frac{\alpha\xi_1+\varphi\xi_2}{1-2(\alpha\xi_1+\varphi\xi_2)})\sqrt{h(t)}\right]\right\}^2\right\}$$

$$z^*_t = \sqrt{1-2(\alpha\xi_1+\varphi\xi_2)}\left[z_t+(\mu+\frac{\alpha\xi_1+\varphi\xi_2}{1-2(\alpha\xi_1+\varphi\xi_2)})\sqrt{h(t)}\right], \tag{A.10}$$

we have the relationship between distribution under the physical measure and risk-neutral measure. The entire risk-neutral process can be given by using (A.10).

B. Conditional Variance and Covariance under Risk-Neutral

Based on equations (4b) and (4c), the risk-neutral multiple volatility components model

$$h^*_t = q^*_t + \tilde{\beta}^*(h^*_{t-1}-q^*_{t-1}) + \alpha^*\left\{\left[z^*_{t-1}-\gamma^*_1\sqrt{h^*_{t-1}}\right]^2-\left[1-2(\alpha\xi_1+\varphi\xi_2)+\gamma_1^{*2}h^*_{t-1}\right]\right\}$$

$$q^*_t = \omega^* + \rho^*q^*_{t-1} + \varphi^*\left\{\left[z^*_{t-1}-\gamma^*_2\sqrt{h^*_{t-1}}\right]^2-\left[1-2(\alpha\xi_1+\varphi\xi_2)+\gamma_2^{*2}h^*_{t-1}\right]\right\},$$

First, the formula of conditional variance of volatility is

$$\mathrm{Var}_{t-1}(h^*_{t+1}) = E_{t-1}[h^*_{t+1}-E_{t-1}(h^*_{t+1})]^2, \tag{B.1}$$

where

$$E_{t-1}(h^*_{t+1}) = E_{t-1}(q^*_{t+1}) + \tilde{\beta}^*(h^*_t-q^*_t) + E_{t-1}\{\alpha^*[(z^*_t-\gamma^*_1\sqrt{h^*_t})^2-(1-2(\alpha\xi_1+\varphi\xi_2)+\gamma_1^{*2}h^*_t)]\};$$

$$E_{t-1}(q^*_{t+1}) = \omega^* + \rho^*q^*_t + E_{t-1}\{\varphi^*[(z^*_t-\gamma^*_2\sqrt{h^*_t})^2-(1-2(\alpha\xi_1+\varphi\xi_2)+\gamma_2^{*2}h^*_t)]\}.$$

Therefore

$$\mathrm{Var}_{t-1}(h^*_{t+1}) = E_{t-1}[h^*_{t+1} - E_{t-1}(h^*_{t+1})]^2$$

$$= E_{t-1}\Big\{\alpha^*(z^*_t - \gamma^*_1\sqrt{h^*_t})^2 + \varphi^*(z^*_t - \gamma^*_2\sqrt{h^*_t})^2$$

$$-E_{t-1}\big[\alpha^*(z^*_t - \gamma^*_1\sqrt{h^*_t})^2\big] - E_{t-1}\big[\varphi^*(z^*_t - \gamma^*_2\sqrt{h^*_t})^2\big]\Big\}^2$$

$$= E_{t-1}\Big\{(\alpha^* + \varphi^*)z^{*2}_t - 2\big(\alpha^*\gamma^*_1 + \varphi^*\gamma^*_2\big)z^*_t\sqrt{h^*_t} - (\alpha^* + \varphi^*)\Big\}^2. \tag{B.2}$$

Using the fact that if z^*_t is a standard normal distribution, $E[z^{*3}_t] = 0$ and $E[z^{*4}_t] = 3$, then

$$\mathrm{Var}_{t-1}(h^*_{t+1}) = 2(\alpha^* + \varphi^*)^2 + 4\big(\alpha^*\gamma^*_1 + \varphi^*\gamma^*_2\big)^2 h^*_t.$$

Second, the conditional correlation of return and volatility under risk-neutral is

$$\mathrm{Cov}_{t-1}(R_t, h^*_{t+1}) = E_{t-1}\{[R_t - E_{t-1}(R_t)][h^*_{t+1} - E_t(h^*_{t+1})]\}, \tag{B.3}$$

where

$$R_t = \ln[S_t/S_{t-1}] = r - \frac{1}{2}h^*_t + \sqrt{h^*_t}z^*_t.$$

And then

$$\mathrm{Cov}_{t-1}(R_t, h^*_{t+1}) =$$

$$E_{t-1}\Big\{\sqrt{h^*_t}z^*_t \cdot \big[(\alpha^* + \varphi^*)z^{*2}_t - 2(\alpha^*\gamma^*_1 + \varphi^*\gamma^*_2)z^*_t\sqrt{h^*_t} - (\alpha^* + \varphi^*)\big]\Big\}.$$

Using the fact that the skewness of standard normal distribution is zero, so that

$$\mathrm{Cov}_{t-1}(R_t, h^*_{t+1}) = -2(\alpha^*\gamma^*_1 + \varphi^*\gamma^*_2)h^*_t.$$

Therefore, the results as following

$$\mathrm{Corr}_{t-1}(R_t, h^*_{t+1}) = \frac{-2(\alpha^*\gamma^*_1 + \varphi^*\gamma^*_2)\sqrt{h^*_t}}{\sqrt{2(\alpha^* + \varphi^*)^2 + 4(\alpha^*\gamma^*_1 + \varphi^*\gamma^*_2)^2 h^*_t}},$$

and the similar way to proof the correlation between return and long-run or short-run volatility which under risk-neutral.

C. Proof of Corollary 2

Recalling the pricing kernel,

$$\frac{M_t}{M_{t-1}} = \left[\frac{S_t}{S_{t-1}}\right]^{\phi} \exp\{\delta + \eta_1(h_t - q_t) + \eta_2 q_t + \xi_1[(h_{t+1} - q_{t+1}) - (h_t - q_t)] + \xi_2[q_{t+1} - q_t]\}, \quad \text{(C.1)}$$

where $R_t = \ln S_t/S_{t-1}$. And the volatility dynamics

$$h_{t+1} = q_{t+1} + \tilde{\beta}(h_t - q_t) + \alpha[(z_t - \gamma_1\sqrt{h_t})^2 - (1 + \gamma_1^2 h_t)] \quad \text{(C.2a)}$$

$$q_{t+1} = \omega + \rho q_t + \varphi[(z_t - \gamma_2\sqrt{h_t})^2 - (1 + \gamma_2^2 h_t)], \quad \text{(C.2b)}$$

Rewriting the return dynamics as

$$R_t = r + \left(\mu - \frac{1}{2}\right) h_t + \sqrt{h_t} z_t,$$

so that

$$z_t = \frac{R_t - r - \left(\mu - \frac{1}{2}\right)h_t}{\sqrt{h_t}}. \quad \text{(C.3)}$$

Substituting (C.2a), (C.2b), and (C.3) into the equation (C.1), we get

$$\ln\left[\frac{M_t}{M_{t-1}}\right] = \frac{(\alpha\xi_1 + \varphi\xi_2)}{h_t}(R_t - r)^2 - \mu(R_t - r) + \frac{1}{2}\left[\left(\mu - \frac{1}{2}\right)^2 - \frac{1}{4[1 - 2(\alpha\xi_1 + \varphi\xi_2)]}\right] h_t - r + \frac{1}{2}\ln[1 - 2(\alpha\xi_1 + \varphi\xi_2)]$$

D. Option Valuation

We follow the option valuation in Heston and Nandi (2000) by using Moment Generating Function (MGF). The risk-neutral dynamic is

$$\ln[S_t] = \ln[S_{t-1}] + r - \frac{1}{2}h_t^* + \sqrt{h_t^*} z_t^*$$

$$h_t^* = q_t^* + \tilde{\beta}^*(h_{t-1}^* - q_{t-1}^*) + \alpha^*\left\{\left[z_{t-1}^* - \gamma_1^*\sqrt{h_{t-1}^*}\right]^2 - \left[1 - 2(\alpha\xi_1 + \varphi\xi_2) + \gamma_1^{*2} h_{t-1}^*\right]\right\}$$

$$q_t^* = \omega^* + \rho^* q_{t-1}^* + \varphi^*\left\{\left[z_{t-1}^* - \gamma_2^*\sqrt{h_{t-1}^*}\right]^2 - \left[1 - 2(\alpha\xi_1 + \varphi\xi_2) + \gamma_2^{*2} h_{t-1}^*\right]\right\},$$

Reference to Christoffersen et al. (2008), the conditional MGF corresponding to this process is

$$g^*_{t,T}(\theta) \equiv E^*_t\{\exp[\theta \ln(S_T)]\}.$$

We guess that the MGF has the log-linear form. We use the more parsimonious notation A_t to indicate $A(t,T;\theta)$ and similarly for $B_{1,t}$ and $B_{2,t}$ and write

$$g^*_{t,T}(\theta) = \exp\left[\theta \ln(S_t) + A_t + B_{1,t}(h^*_{t+1} - q^*_{t+1}) + B_{2,t}q^*_{t+1}\right], \qquad \text{(D.1)}$$

Because the stock price in time T has known. The terminal condition is

$$A_T = B_{1,T} = B_{2,T} = 0.$$

Next we extend equation (B.2) to time $t+1$

$$g^*_{t+1,T}(\theta) = E^*_t\left\{\exp\left[\theta \ln(S_{t+1}) + A_{t+1} + B_{1,t+1}[h^*_{t+2} - q^*_{t+2}] + B_{2,t+1}q^*_{t+2}\right]\right\},$$

and then

$$g^*_{t+1,T}(\theta) = E^*_t\left\{\exp\left[\theta\left(\ln[S_t] + r - \frac{1}{2}h^*_{t+1} + \sqrt{h^*_{t+1}}z^*_{t+1}\right) + A_{t+1} + B_{1,t+1}\cdot\right.\right.$$

$$(\tilde{\beta}^*(h^*_{t+1} - q^*_{t+1}) + \alpha^*(\left(z^*_{t+1} - \gamma^*_1\sqrt{h^*_{t+1}}\right)^2 - (1 - 2(\alpha\xi_1 + \varphi\xi_2) +$$

$$\gamma^{*2}_1 h^*_{t+1}))) + B_{2,t+1}(\omega^* + \rho^* q^*_{t+1} + \varphi^*((z^*_t - \gamma^*_2\sqrt{h^*_{t+1}})^2 - (1 -$$

$$\left.\left.2(\alpha\xi_1 + \varphi\xi_2) + \gamma^{*2}_2 h^*_{t+1})))\right]\right\}$$

$$= E^*_t\left\{\exp\left[\theta(\ln(S_t) + r) + A_{t+1} - \frac{1}{2}\theta h^*_{t+1} + B_{1,t+1}\tilde{\beta}^*(h^*_{t+1} - q^*_{t+1}) +\right.\right.$$

$$B_{2,t+1}(\omega^* + \rho^* q^*_{t+1}) - \left(1 - 2\left(\alpha\xi_1 + \varphi\xi_2\right)\right)\left(B_{1,t+1}\alpha^* + B_{2,t+1}\varphi^*\right) +$$

$$2z^*_{t+1}\sqrt{h^*_{t+1}}\left(\frac{1}{2}\theta - \alpha^*\gamma^*_1 B_{1,t+1} - \varphi^*\gamma^*_2 B_{2,t+1}\right) + \left(\alpha^* B_{1,t+1} +\right.$$

$$\left.\left.\left.\varphi^* B_{2,t+1}\right)z^{*2}_{t+1}\right]\right\}.$$

We also use the formula which also used in Appendix A for standard normal

$$E[\exp\{az^2 + 2abz\}] = \exp\{-\frac{1}{2}\ln(1-2a) + \frac{2a^2b^2}{1-2a}\}.$$

Let

$$a = \alpha^* B_{1,t+1} + \varphi^* B_{2,t+1};$$

$$b = \frac{\left(\frac{1}{2}\theta - \alpha^*\gamma^*_1 B_{1,t+1} - \varphi^*\gamma^*_2 B_{2,t+1}\right)}{\alpha^* B_{1,t+1} + \varphi^* B_{2,t+1}}\sqrt{h^*_{t+1}},$$

then

$$2ab = 2\left(\frac{1}{2}\theta - \alpha^*\gamma_1^* B_{1,t+1} - \varphi^*\gamma_2^* B_{2,t+1}\right)\sqrt{h_{t+1}^*}$$

so that

$$E[\exp\{az^2+2abz\}] =$$

$$\exp\left\{-\frac{1}{2}\ln[1-2(\alpha^* B_{1,t+1}+\varphi^* B_{2,t+1})] + \frac{2\left(\frac{1}{2}\theta-\alpha^*\gamma_1^* B_{1,t+1}-\varphi^*\gamma_2^* B_{2,t+1}\right)^2 h_{t+1}^*}{1-2(\alpha^* B_{1,t+1}+\varphi^* B_{2,t+1})}\right\}.$$

Therefore, we can get

$$g_{t+1,T}^*(\theta) = E_t^*\Bigg\{\exp\Bigg[\theta(\ln S_t) + \theta r + A_{t+1} - \frac{1}{2}\ln[1-2(\alpha^* B_{1,t+1}+\varphi^* B_{2,t+1})] - [1-2(\alpha\xi_1+\varphi\xi_2)](B_{1,t+1}\alpha^* + B_{2,t+1}\varphi^*) + B_{1,t+1}\tilde{\beta}^*(h_{t+1}^* - q_{t+1}^*) + B_{2,t+1}(\omega^* + \rho^* q_{t+1}^*) - \frac{1}{2}\theta h_{t+1}^* + \frac{2\left(\alpha^*\gamma_1^* B_{1,t+1}+\varphi^*\gamma_2^* B_{2,t+1}-\frac{1}{2}\theta\right)^2}{1-2(\alpha^* B_{1,t+1}+\varphi^* B_{2,t+1})} h_{t+1}^*\Bigg]\Bigg\}. \tag{D.2}$$

The matching term in equations (D.1) and (D.2) gives the results as following

$$A_t = A_{t+1} + r\theta + \omega^* B_{2,t+1} - \frac{1}{2}\ln[1-2(\alpha^* B_{1,t+1}+\varphi^* B_{2,t+1})] - [1-2(\alpha\xi_1 + \varphi\xi_2)][\alpha^* B_{1,t+1} + \varphi^* B_{2,t+1}];$$

$$B_{1,t} = \tilde{\beta}^* B_{1,t+1} - \frac{1}{2}\theta + \frac{2\left(\alpha^*\gamma_1^* B_{1,t+1}+\varphi^*\gamma_2^* B_{2,t+1}-\frac{1}{2}\theta\right)^2}{1-2(\alpha^* B_{1,t+1}+\varphi^* B_{2,t+1})};$$

$$B_{2,t} = \rho^* B_{2,t+1} - \frac{1}{2}\theta + \frac{2\left(\alpha^*\gamma_1^* B_{1,t+1}+\varphi^*\gamma_2^* B_{2,t+1}-\frac{1}{2}\theta\right)^2}{1-2(\alpha^* B_{1,t+1}+\varphi^* B_{2,t+1})}.$$

As we get the MGF and risk-neutral dynamic. Option values can now be computed using

$$C(S_t, h_{t+1}^*, X, T) = S_t P_{1,t} - X\exp[-r(T-t)]P_{2,t},$$

where

$$P_{1,t} = \frac{1}{2} + \frac{1}{\pi}\exp[-r(T-t)]\int_0^\infty \mathrm{Re}\left[\frac{X^{-i\theta} g_{t,T}^*(i\theta+1)}{i\theta S_t}\right] d\theta$$

$$P_{2,t} = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{X^{-i\theta} g_{t,T}^*(i\theta)}{i\theta}\right] d\theta.$$

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