Improving Nelson-Siegel term structure model under zero / super-low interest rate policy

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Abstract

In this paper, we explored improvement on Nelson-Siegel model under zero / super-low interest rate policy. We tried to understand economical meanings of the model by assuming a stochastic process that NS model be derived. And we naturally extended the model in order to capture the full term structure regulated its short-end (spot rate) to be zero. Furthermore, we explored the expressive ability of NS model by considering coupon rate effect and JGB future effect. As the results, unnatural irregularity and negative value observed in short-term range have reduced and explanatory power of extended NS model has improved.

1. Introduction

Nelson and Siegel (1987) represented cross-sectional term structure model, known as Nelson-Siegel model (hereafter NS model) have been widely used so far, because of its simplicity and convenient structure for variety of applications. NS model is shown as a linear combination of simple three factors corresponding to level, slope and curvature. It provides a convenient framework to examine the relationship between macroeconomic variables and determinants of the term structure, and to consider the attitude towards future interest rate of market participants (Fujii and Takaoka (2008)). Furthermore, It has been applied to the evaluation of volatility risk and future forecast by modeling the dynamic time-series structure behind each factor (Diebold and Li (2006)). On the other hand, no-arbitrage conditions on NS model has also been studied (Filipovic(2009)), it is not reached the stage to practical application, since the complex mathematical consequence spoils the advantage of easy-to-understand format.

There are several empirical researches applying NS model on Japanese government bond (JGB). Kikuchi and Shintani (2012) pointed out that NS model was not appropriate to be applied to JGB market because it might show negative interest rate and abnormal shape in short term region. In this background, it is presumed that the term structure, especially the short-term region, has been significantly affected by prolongation of zero / ultra-low interest rate policy. In addition, there are also indications that the bond-specific differences such as coupon rate, issue amount, trading amount and derivative related transactions have negligible impact. (Kariya et al. (2013)).

In this paper, we investigate economical meanings of NS model by assuming a stochastic process that the model be derived. And we also investigate the expressive ability of NS model by considering individual bonds attributions such as coupon rate effect and JGB future effect. Through the collapse of Lehman Brothers and the Greek debt crisis, the zero / super-low interest rate policy were also underway in many countries. Therefore, proposing a sufficiently usable NS model in such interest rate situation must be important.

2. Nelson-Siegel model

(1) Overview of NS model

Cross-sectional term structure model proposed in Nelson and Siegel (1987) is defined as a continuous forward rate of maturity x year:

$$f(x) = \beta_0 + \beta_1 \exp\left(-\frac{x}{\tau_1}\right) + \beta_2 \frac{x}{\tau_1} \exp\left(-\frac{x}{\tau_1}\right).$$

Here, $\beta_0, \beta_1, \beta_2, \tau_1$ are parameters usually estimated from bond prices data or term structure of certain maturity term points. The first term β_0 refers to a parallel component which does not depend on maturity, and the second term $\beta_1 \exp\left(-\frac{x}{\tau_1}\right)$ which is monotonically decreasing function on maturity x means slope component, and the third term $\beta_2 \frac{x}{\tau_1} \exp\left(-\frac{x}{\tau_1}\right)$ which is single-peak function represents the curvature component of the term structure. Svensson (1995) proposed the following model by adding the fourth term into the original NS model aiming to enhance the traceability of the curve component:

$$f(x) = \beta_0 + \beta_1 \exp\left(-\frac{x}{\tau_1}\right) + \beta_2 \frac{x}{\tau_1} \exp\left(-\frac{x}{\tau_1}\right) + \beta_3 \frac{x}{\tau_2} \exp\left(-\frac{x}{\tau_2}\right) .$$

In this paper, we refer this model as SV model in distinction from NS model, however, we sometimes regard them as NS model in a broad sense.

Zero yield $y(x)$ can be calculated from the forward rates follows:

$$y(x) = \frac{1}{x}\int_0^x f(u)du$$

Then the term structure of zero yield, which is derived from SV model and NS model, are as follows.

$$y(x) = \beta_0 + \beta_1 \left\{\frac{1 - \exp(-x/\tau_1)}{x/\tau_1}\right\} + \beta_2 \left\{\frac{1 - \exp(-x/\tau_1)}{x/\tau_1} - \exp(-x/\tau_1)\right\}$$

$$y(x) = \beta_0 + \beta_1 \left\{\frac{1-\exp(-x/\tau_1)}{x/\tau_1}\right\} + \beta_2 \left\{\frac{1-\exp(-x/\tau_1)}{x/\tau_1} - \exp(-x/\tau_1)\right\} + \beta_3 \left\{\frac{1-\exp(-x/\tau_2)}{x/\tau_2} - \exp(-x/\tau_2)\right\}$$

(2) Method of estimating the models

To estimate the model parameters from bond price data, it is usually necessary to solve a mathematical programming problem which minimizes the total difference between market prices and theoretical prices determined by the model. Specifically, given the initial value of the parameter as $\theta = \{\beta_0, \beta_1, \beta_2, \tau_1\}$ and zero yield of NS model $y(x|\theta)$, theoretical price of the i-th bond which has cash flow $c_i(t_j),\ j = 1, \dots, n_i$ is as follows:

$$P_i^* = \sum_{j=1}^{n_i} c_i(t_j)\ \exp(-y(t_j)t_j),$$

By solving a mathematical programming problem, such as to minimize the sum of squared error of the theoretical price from the market price across the bonds traded in the market, the parameter θ of the NS model can be determined.

$$\min_{\boldsymbol{\theta}} \sum_{i=1}^{N} w_i (P_i - P_i^*)^2$$

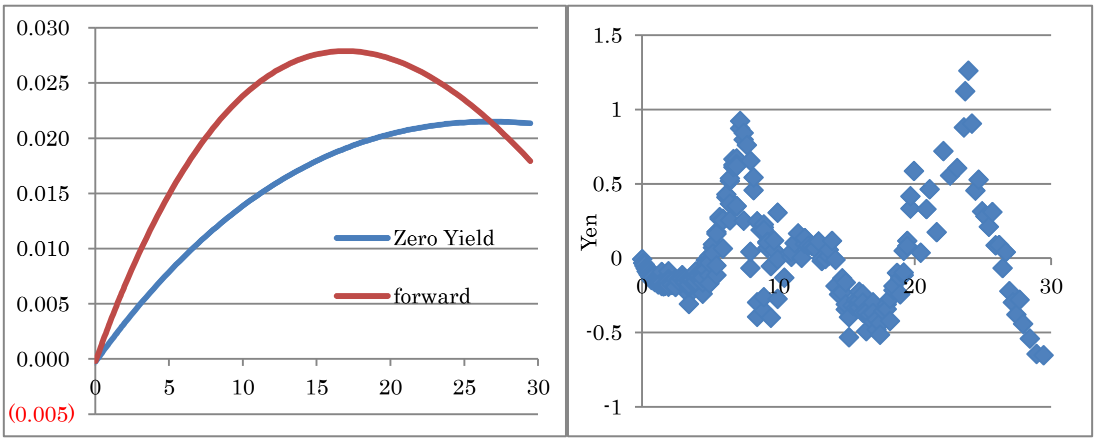
Here, P_i is the market price of i-th bond, w_i is suitable weight term. In this paper, we set $w_i = 1$ according to Kikuchi and Shintani (2012).

2. Improving NS model by focusing on individual bond attribution

(1)Application case of NS model

We show estimation results of SV model according the procedure in preceding section as preliminary study. We use fixed rate JGB price data at March 31, 2009 downloaded from the Japan Securities Dealers Association website.

Fig.1　Term structure estimated by SV model and estimation error of each bond



The estimation errors (yen) are plotted by each bond with respect to its maturity.

Estimated zero yield and forward rate are smooth enough but the pricing errors show irregularity depending on the maturity, then the expressive power of the model seems lacking. However, instead of choosing the easy solutions by piecewise polynomial such as spline function to estimate the curve less uneven, we think it should be more valuable to consider the reason why such errors have occurred. For example, the first peak of the pricing errors arise near 7 year maturity, since it is a term of the cheapest government bond can be settled with futures contract exist, so we do consider the relation with government bond futures.

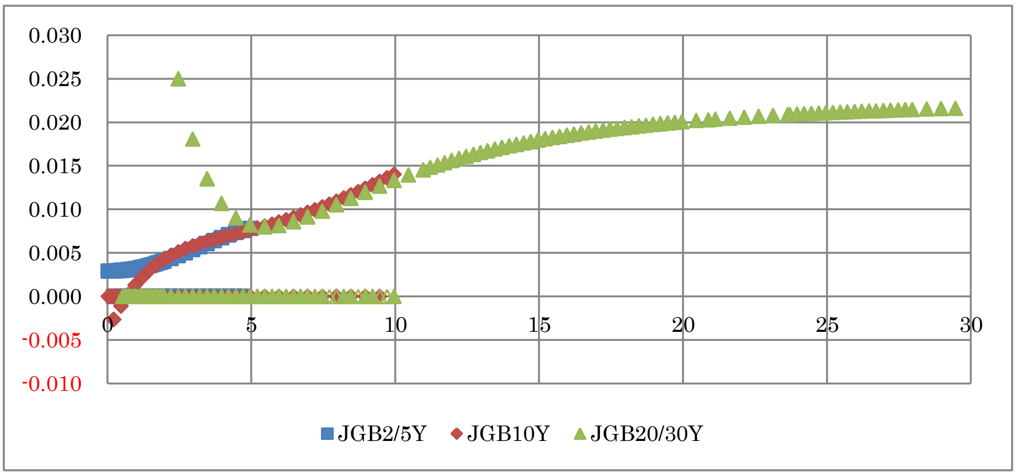
Also, because the peak of between 20 and 30 year depends on the ultra-long-term bonds, we should confirm existing excess demand for ultra-long-term bonds which have relatively high coupons (coupon effect), or the presence of investors who have higher preference to ultra-long-term bonds (market disruption effect). There seems to need considering on impact of issue amount and elapsed years after issuance (market liquidity), and the reliability of the price of the bonds that are not traded in the most imminent redemption.

(2) Examination of individual attributes effect

We divided the bonds into three groups according to the bond maturity, ①2 / 5 year, government bonds ②10 years, and ③20 / 30 years, and estimated SV model for each group. Fig. 2 shows the results.

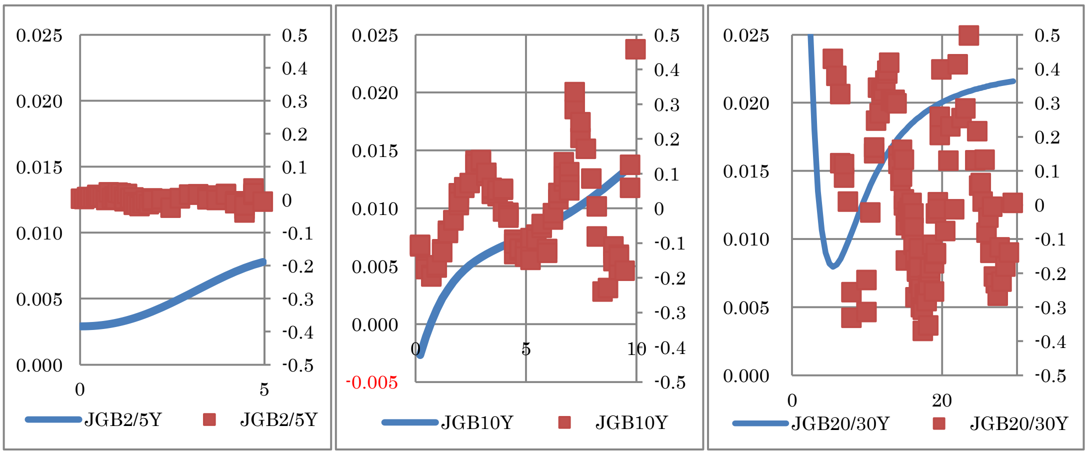
If you look at the term structure, large deviation due to the difference in bond type is observed, such as less than five years range of 20/30-year bonds, less than two years range of 10 year and 2/5 year bonds.

Fig.2　Estimation results by bond type



Looking at the error distribution for each bond types (Fig. 3), since the error looks to be dependent on the maturity, it is concern that the potential descriptive power of SV model can be insufficient. In other words, it is possible to understand that additional factor can be required for explaining the term structure. In addition, for the error that has been estimated from each of 10-year and 20/30 year JGB, there are regions that errors increase locally. About 10-year results, around 7 years is the region that the cheapest-to-deliver interchangeable with government bond futures exists. About 20/30 year results, less than five years is the region that 5% or more coupon bonds exist. Thus, we should consider the following additional factors if available.

Fig.3　Estimated error distribution for different bond type



① Coupon effect

High coupon bonds are preferable because of tax effect. We assume simple relation between coupon and discount rate in this paper.

② Bond type effect

In order to absorb the impact of differences between the types of bonds on the prices, dummy variables, for example 5-year, 10-year and 20/30-year JGB can be good proxy. But in this study, because we are going to introduce coupon effect variables which seems have significant correlation to bond type, we don't hire bond type dummy variables.

③ JGB future effect

Conversion factor for 10-year JGB which can be settled for 10-year JGB future, and we introduce them as adjustment factor.

④ Liquidity effect

Since it is difficult to measure the liquidity without trading data (we cannot access trading volume, bid-ask spread.) , proxy variables can be hired for the analysis, such as issue amount of each individual bonds, the frequency of price changes in a certain business days, elapsed years after issuance and remaining period to maturity.

In this paper, we preliminary verified issue amount and frequency of price changes as proxy for liquidity, but there seemed no explanatory power for those variables, we decided to postpone considering liquidity effect.

(3) Conditions on the zero interest rate policy

As zero / super-low interest rate policy has been carried out over an extended period in Japan, short-term end points of the interest rate term structure had been anchored to almost zero. In order to reflect these conditions explicitly in the model, the short-term endpoint is required to be 0. That is,

$$f(0) = \beta_0 + \beta_1 = 0$$

for both NS and SV model. Furthermore, under zero interest rate situations, the slope in the short-term end points of the term structure can be non-negative because the spot rate won't be able to go down any more. That is, NS model should require following constraint:

$$\left.\frac{df(x)}{dx}\right|_{x=0} = \frac{\beta_2 - \beta_1}{\tau_1} \geq 0$$

And for SV model,

$$\left.\frac{df(x)}{dx}\right|_{x=0} = \beta_2 - \beta_1 + \frac{\tau_1}{\tau_2}\beta_3 \geq 0$$

This constraint for each model impose a restriction on the parameters, hence they cannot be determined independently.

If we estimated NS/SV model under zero rate situation regardless whether explicitly introduced these constraints or not, it should be noted that problems are likely to arise in the interpretation of each original parameter (level, slope, curvature) when we try to fit the model precisely to short-range term structure shape. Conversely, it may be necessary to consider how to increase the degree of freedom of NS / SV model because they cannot fully reflect the impact of the zero interest rate policy.

3. Economical understanding and extension of NS model

We fix the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\{t_0, t_1, t_2, \cdots\}$ be a sequence of non-negative random variables satisfying $t_0 = 0,\ t_{i-1} < t_i,\ i = 1 \cdots,$ we call this point process on $\mathbb{R}_+$. When the intervals of the point process $\mathrm{d_i} = t_i - t_{i-1}, i = 1, \cdots$ follow independent exponential random variable with intensity λ, then, the right continuous stochastic process defined by the following equation is a Poisson process :

$$N(t) = \sum_{k=1}^{\infty} 1_{\{t_k \le t\}},$$

And the probability distribution is as follows.

$$\mathbb{P}\{N(t) = k\} = \frac{(\lambda t)^k}{k!} \mathrm{e}^{-\lambda \mathrm{t}}, \quad \mathrm{k} = 0, 1, \ldots$$

If we accept the pure expectation hypothesis that require no term risk premium, the forward rate term structure can be determined as expected value of future trajectory of spot rate. So, we assume the spot rate process follows a compound Poisson process, and then the forward rate can be expressed as follows.

$$f(t) = r_0 + \mathbb{E}_0\left[\sum_{k=1}^{N(t)} J_k\right] = r_0 + \mathbb{E}_0\left[\mathbb{E}_t[\ J_1 + J_2 + \cdots + J_{N(t)} \mid N(t)\ ]\right],$$

Here r_0 is initial spot rate at time t=0, J_k is a random variable representing the k-th jump width, and $\mathbb{E}_t[\,]$ is expected value operator conditioned on at time t. For the sake of simplicity, it is further assumed that J_k and $N(t)$ is independent, Let $\mathbb{E}_0[J_1 + \cdots + J_k\,] = \eta_k$, the forward rate is possible to represent as follows.

$$\begin{aligned} f(t) &= r_0 + \mathbb{E}_0[J_1\,]\, \mathbb{E}_0\left[1_{\{N(t)=1\}}\right] + \cdots + \mathbb{E}_0[J_1 + \cdots + J_n\,]\, \mathbb{E}_0\left[1_{\{N(t)=n\}}\right] + \cdots \\ &= r_0 + \eta_1\, \mathbb{P}[N(t) = 1\,] + \cdots + \eta_n \mathbb{P}[N(t) = n\,] + \cdots \\ &= r_0 + \eta_1\, \lambda t\, e^{-\lambda t} + \cdots + \eta_n \frac{(\lambda t)^n}{n!} e^{-\lambda t} + \cdots \end{aligned}$$

However, in order to estimate the actual term structure, it is necessary to finite the number of jumps (factors). Therefore, we consider n-factor forward rate model :

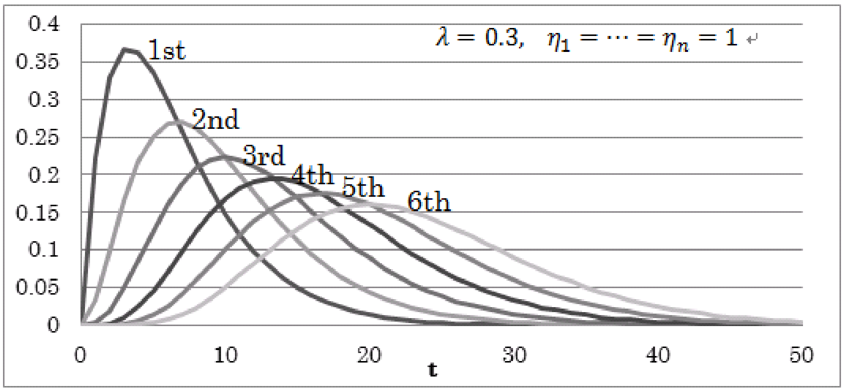
$$f_n(t) = r_0 + \eta_1 \lambda t\, e^{-\lambda t} + \cdots + \eta_n \frac{(\lambda t)^n}{n!} e^{-\lambda t}$$

Each factor of this model shows uni-modal distribution (see Fig. 4), and it satisfies

$$\lim_{t \to \infty} f_n(t) = r_0 .$$

Therefore, in order to replicate the entire term structure accurately up to the maximum maturity (30 years, for example), it is necessary to add sufficient number of uni-modal shape factors by controlling the shape parameter λ which governs the peak of of each factor.

Fig.4 the shape of factors



Now let $\lambda = \frac{1}{\tau}$, $\frac{\eta_n}{n!} = \beta_n$, then n-factor forward rate becomes

$$f_n(t) = r_0 + \beta_1 \frac{t}{\tau} e^{-\frac{t}{\tau}} + \beta_2 \left(\frac{t}{\tau}\right)^2 e^{-\frac{t}{\tau}} + \cdots + \beta_n \left(\frac{t}{\tau}\right)^n e^{-\frac{t}{\tau}} .$$

This model class cannot include NS model. NS model is obtained by adding factor $\beta_0 e^{-\frac{t}{\tau}}$ to $f_1(t)$,

$$f_{NS}(t) = f_1(t) + \beta_0 e^{-\frac{t}{\tau}} = r_0 + \beta_0 e^{-\frac{t}{\tau}} + \beta_1 \frac{t}{\tau} e^{-\frac{t}{\tau}} ,$$

or adding the same one into $f_2(t)$ then we get Svensson (SV) model (actually, SV has two different shape parameters τ_1 and τ_2).

$$f_{SV}(t) = f_2(t) + \beta_0 e^{-\frac{t}{\tau}} = r_0 + \beta_0 e^{-\frac{t}{\tau}} + \beta_1 \frac{t}{\tau} e^{-\frac{t}{\tau}} + \beta_2 \left(\frac{t}{\tau}\right)^2 e^{-\frac{t}{\tau}}$$

The $\beta_0 e^{-\frac{t}{\tau}}$ factor in the NS / SV model is monotonic function and converges to β_0 ($t \to 0$), and to 0 ($t \to \infty$). So,

$$f_{NS}(t) = \begin{cases} r_0 + \beta_0 , & t = 0 \\ r_0 , & t \to \infty \end{cases}$$

It is reasonable to say that β_0 is yield spread and $\beta_0 e^{-\frac{t}{\tau}}$ can naturally reflect the term premium if we abandon the expectation hypothesis.[1]

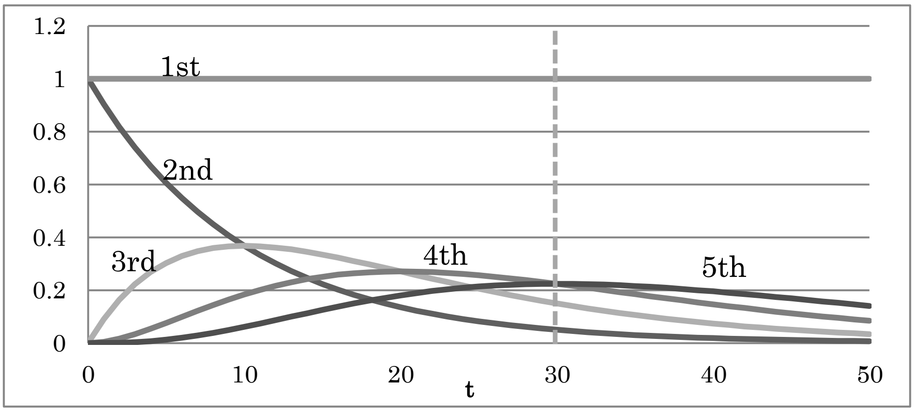
Now we can point out a couple of comments on NS model as follows.

Since the shape parameter $\tau(=1/\lambda)$ is defined as the reciprocal of Poisson jump intensity, τ must have relationship with β_i which dominate jump width of each i-th jump, that is, when the jump intensity increase, the jump width must be decrease to capture the actual expectation of forward rate trajectory. Once we accept the trade-off relation between jump intensity τ and jump width β_i, we should pre-determine suitable size of τ in accordance with the number of factors instead of estimating τ and β_i simultaneously. The difficulty with estimating parameters of NS/SV model repeatedly mentioned in previous researches can be caused by the dependencies between those parameters.

For example, extended NS model with additional 2 factors associated with the expected value of cumulative jumps is as follows.

$$f_{NS}^{(5)}(t) = \beta_0 + \beta_1 e^{-\frac{t}{\tau}} + \beta_2 \frac{t}{\tau} e^{-\frac{t}{\tau}} + \beta_3 \left(\frac{t}{\tau}\right)^2 e^{-\frac{t}{\tau}} + \beta_4 \left(\frac{t}{\tau}\right)^3 e^{-\frac{t}{\tau}}$$

Fig.5 the shape of factors, τ=10



For this 5-factor model (including constant β_0 as a factor), τ=10 seems to be a suitable candidate to represent the full term structure up to maturity around 30 years. See Fig.5, it is apparent relative differences between the 5 factors within 50 years, while the high-order factors have similar shapes in the region of 40-50 years. Thus, τ dominates the shape of each factor and we should determine it with considering the number of employed factors.

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[1] In this case, β_0 could be negative so that $\beta_0 e^{-\frac{t}{\tau}}$ become increasing function with respect to maturity t.

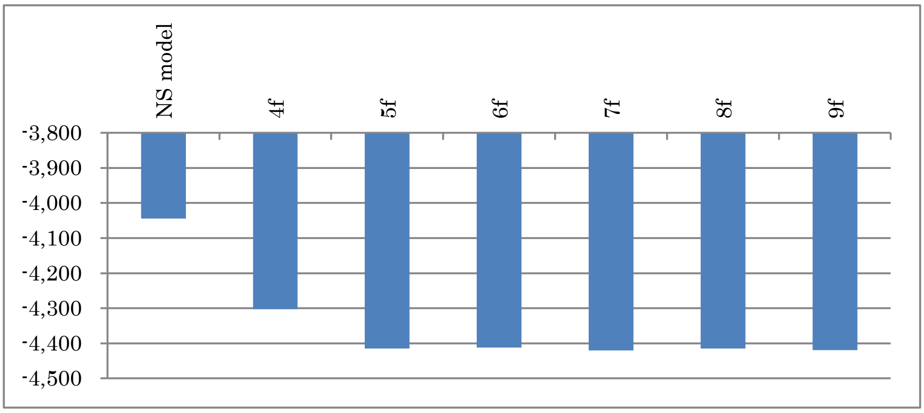
4. Estimation

(1)Pre-determined number of factors

We determine the most optimal number of factors for considering dataset by AIC of extended NS model. As a result, 5-factor model which is identical to extended NS model with two additional factors are selected .

$$f_{NS}^{(5)}(t) = \beta_0 + \beta_1 e^{-\frac{t}{\tau}} + \beta_2 \frac{t}{\tau} e^{-\frac{t}{\tau}} + \beta_3 \left(\frac{t}{\tau}\right)^2 e^{-\frac{t}{\tau}} + \beta_4 \left(\frac{t}{\tau}\right)^3 e^{-\frac{t}{\tau}}$$

Fig.6 AIC (averaged value from 2006/9 – 2011/7)



(2) model

In this paper, we estimate extended NS model that takes into account coupon effect, government bond futures effect. We show the procedures in case of 5 factor model ($f_{NS}^{(5)}(t)$). First, the objective function is as follows.

$$\min_{\boldsymbol{\theta}} \sum_{i=1}^{N} (P_i - P_i^*)^2$$

θ means parameter set: $\theta = \{\beta_0, \beta_1, \beta_2, \beta_3, \beta_4, \tau, \delta_{10}, \delta_{20}, \delta_{30}, \gamma_{CV}\}$, and theoretical bond price is assumed to be :

$$P_i^* = \sum_{j=1}^{n_i} c_i(t_j) \exp[-\{y(t_j)t_j + \delta_{10}C_{10,i} + \delta_{20}C_{20,i} + \delta_{30}C_{30,i}\} t_i] + \gamma_{CV}CV_i$$

Here, $c_i(t_j)$ is a cash flow at time t_j of i-th bond, $C_{m,i}$ is standardized (mean=0, s.d.=1) coupon rate of i-th bond in each bond maturity type (initial maturity), and $D_{m,i}$ is dummy variable for i-th bond according to the current maturity. CV_i is reciprocal of conversion factor for i-th bond (if available). [2] $y(t_j)$ is zero-yield term structure derived

[2] The lowest conversion factor means the cheapest bond. Hence, by taking the inverse, It is expect that the largest CV bond will be the most vulnerable to demand/supply change of futures.

from the forward rate model as follows:

$$y(x) = \beta_0 + \beta_1 \left\{ \frac{1-\exp(-x/\tau)}{x/\tau} \right\} + \beta_2 \left\{ \frac{1-\exp(-x/\tau)}{x/\tau} - \exp(-x/\tau) \right\}$$
$$+ 2\beta_3 \left\{ \frac{1-\exp(-x/\tau)}{x/\tau} - \left(\frac{x}{2\tau} + 1 \right) \exp(-x/\tau) \right\}$$
$$+ 6\beta_4 \left\{ \frac{1-\exp(-x/\tau)}{x/\tau} - \left(\frac{x^2}{6\tau^2} + \frac{x}{2\tau} + 1 \right) \exp(-x/\tau) \right\}$$

(3)Data

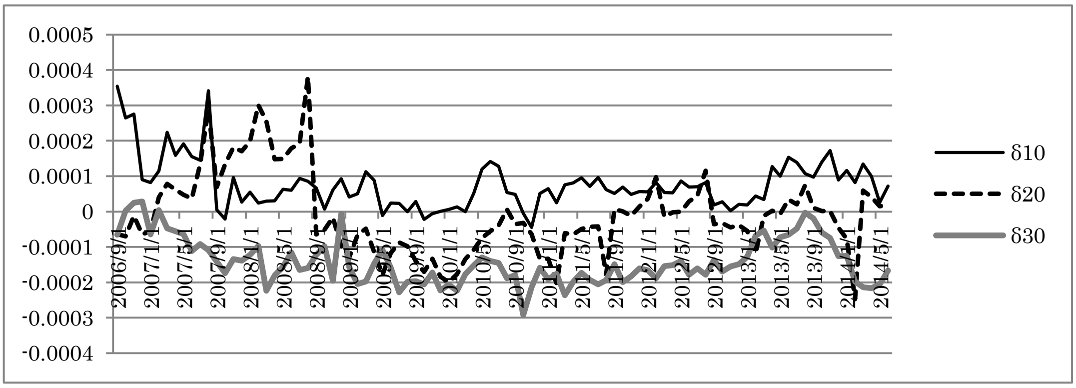
JGB price data used in this research is OTC reference value which is aggregated prices reported from security companies. Those are openly available on web site of the Japan Securities Dealers Association. The universe contains 2, 5, 10, 20, 30-year Japanese government bonds, from 2006 / 9-2011 / 7, (end of month, 59 months).

5. Estimation results

(1) Coupon effect

See Fig. 7, there is a consistent trend in 30-year maturity bonds containing relatively many high-coupon bonds. Since coupon effect of 30-year bonds is observed as negative values, that is the discount rate for high-coupon bonds is relatively low, ie. coupon premium is reflected in the prices.

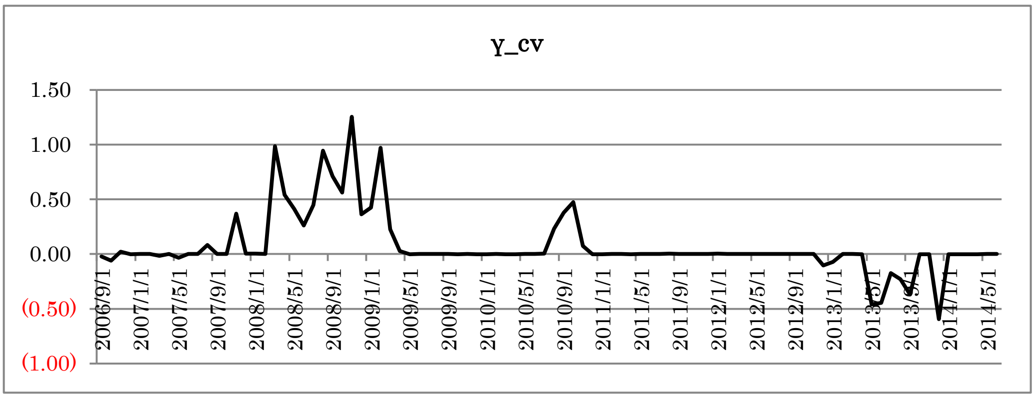
Fig.7 Coupon effect: δ_{10}, δ_{20}, δ_{30}



(5) JGB future effect

From Fig. 8, we can understand that the parameter γ_{CV} suddernly increased in 2008 (Lehman shock) and 2010 (Greec shock) and negatively shocked 2013 (Abenomics: launching a new phase of monetary easing). This factor has been estimated for reciprocal of conversion factor of JGB future, so positive value means that the pricing is more expensive than the theoretical price derived from term structure. The reason why the price of JGB which can be settled with future became more expensive would be a result that demand for long-term government bond futures has increased to noticeable amount due to "flight to quality". On the other hand, in 2013, Bank of Japan launched a new phase of monetary easing

Fig.8　JGB future effect:　γ_{CV}



(6) Estimation results of term structure of zero-yield

By introducing coupon effect and future effect variable, unreasonable irregularity observed on term structure of zero-yield seems to be relaxed (see Fig.9). However, it cannot be sufficiently improved.

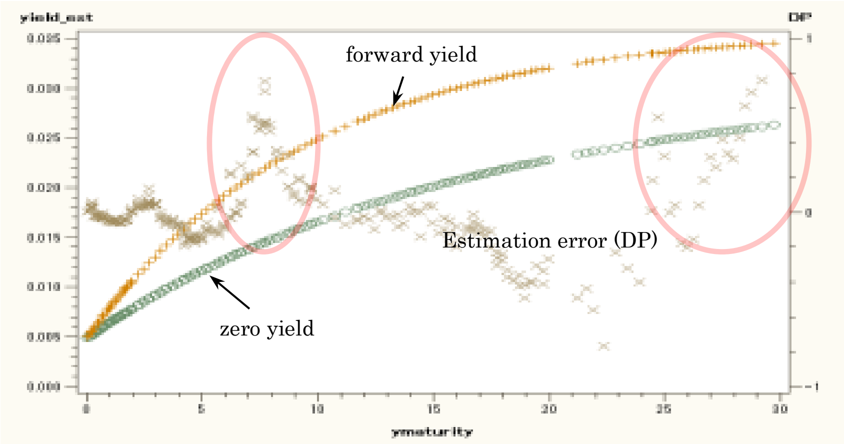
6. Conclusion

In this paper, we explored improvement on NS model under zero interest rate policy, and examined necessary conditions in order to capture the term structure regulated the short spot rate to be zero. Furthermore, we explored the expressive ability of NS model by considering coupon rate effect and JGB future effect. As the results, unnatural irregularities and negative value observed in short-term range have reduced, and explanatory power of NS model has improved (error has reduced) compared to original model).

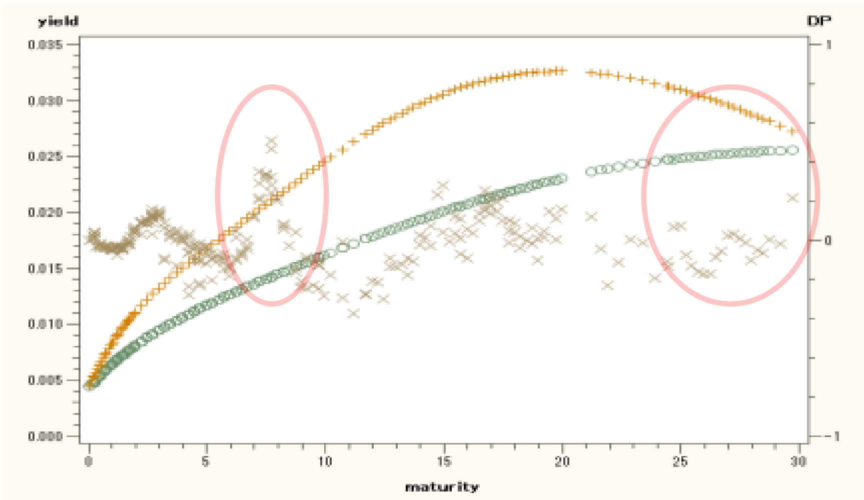
As for individual bond attributes, JGB future effect observed as distorted price of JGBs that exchangeable to JGB-future can be well captured by reciprocal of their conversion factor values, and this adjustment contributed to correct distortion of the term structure.

Fig.9 Term structure of zero-yield (2008/6)

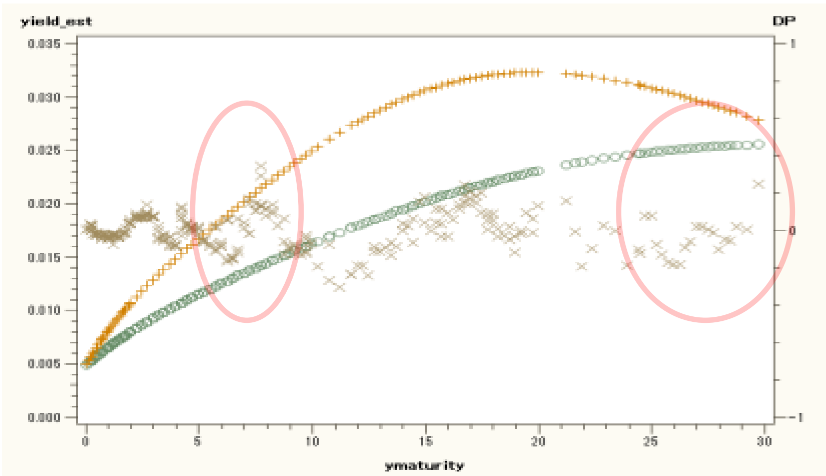
a. 3-factor model (NS model)



b. 5-factor model, no coupon/future variables



c. 5-factor model with coupon/future variables



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