RISK-SHIFTING AND OPTIMAL ASSET ALLOCATION IN LIFE INSURANCE: THE IMPACT OF REGULATION

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Abstract. In a typical participating life insurance contract, the insurance company is entitled to a share of the return surplus as compensation for the return guarantee granted to policyholders. This call-option-like stake gives the insurance company an incentive to increase the riskiness of its investments at the expense of the policyholders. This conflict of interests can partially be solved by regulation deterring the insurance company from taking excessive risk. We suggest that the regulator implement a traffic light system where distressed companies are forced to reduce the riskiness of their asset allocation. In a utility-based framework, we show that this approach can increase the benefits of policyholder and insurance company. At the same time, default probabilities (and thus solvency capital requirements) can be reduced.

Keywords: regulation, life insurance, default risk, utility maximization, risk sharing, multiobjective optimization
JEL: G11, G23

1. Introduction

Participating life insurance contracts usually provide a yearly or maturity guarantee for the policyholders. The surplus above this guaranteed amount is shared between policyholders and the owners (shareholders) of the insurance company. In return, the policyholders pay insurance premiums that are invested by the insurance company. The stakes of policy and shareholders are usually modeled by a contingent-claim approach. The call-option-like stake of shareholders and limited liability gives the insurance company incentives to invest the premiums as riskily as possible at the expense of the policyholder (see, e.g., Doherty and Garven [1986], Cummins [1988]). A very similar conflict arises between debt and equity holder of corporations: Especially if the corporation is in distress, the equity holders tend

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to take on as much risk as possible, a line of action called “risk shifting”. One possibility to solve this conflict of interests is the introduction of a regulator that restricts excessive risk taking of the insurance company (see, e.g., Gatzert and Schmeiser [2008], Dong et al. [2014], Filipović et al. [2015]). The aim of this paper is to propose ways on how such a regulatory scheme can be implemented in order to solve or at least alleviate this conflict of interests.

Regulatory supervision is necessary and justified, as insurance markets are still rather intransparent: An information asymmetry between insurance company and policyholders further increases the described conflict of interests (see, e.g., Rees et al. [1999]). Regulatory supervision in Europe has transformed from relatively simple methods to a comprehensive and very detailed line of action adequately reflecting all the risks inherent in life insurance companies. Thereby, a risk-based supervision relying on Value-at-Risk or default probability gains more and more importance (see, e.g., Bauer et al. [2005]). There are many different designs of regulatory supervision: The regulator may enforce price constraints by introducing restrictions on premium calculation (see, e.g., MacMinn and Witt [1987]). Furthermore, the insurance company may be forced to provide risk-based capital as, for example, specified in the Solvency II accord. The amount of risk capital needed is usually defined to impose an upper bound on default probability (see, e.g., McCabe and Witt [1980]). Related to this, the regulator may impose constraints on the riskiness of the insurance company's investment decisions.[1] For distressed companies, the last is easier to implement than the risk-based provision of capital, because a distressed company might face problems acquiring new capital to fulfill solvency requirements.

Related to this study, the conflict of interests between policyholder and insurance company can be mitigated by a suitable design of the insurance contracts. Several authors analyzed what type of guarantee or securitization meachanism best suits the needs of policy and shareholders (see, e.g., Døskeland and Nordahl [2008], Dong et al. [2014], Schlütter [2014], Filipović et al. [2015]). They commonly assume that the premiums are invested in a portfolio that follows Lévy dynamics, i.e. returns in subsequent periods are independent and identically distributed (typical asset models used are geometric Brownian motion or Merton jump-diffusion,

[1]This might include a restriction of the share of stocks and other risky investments or some minimum diversification requirement. In Germany this is, for example, regulated by §3 Anlageverordnung (AnlV). An overview of regulations in other European countries is given by Davis [2001].

see, e.g., Døskeland and Nordahl [2008], Gatzert and Schmeiser [2008], Schmeiser and Wagner [2013], Dong et al. [2014], Schlütter [2014], Filipović et al. [2015] and many others). However, the option-like and non-linear payoffs of participating life insurance contracts suggest that investing in a constant-risk portfolio is no longer maximizing the benefits of policy and shareholders. This fact was partially recognized in asset-liability management[2]. We suggest that the regulator enforces a non-Lévy, regime-dependent investment strategy in order to alleviate the aforementioned conflict of interests between policy and shareholders. The main idea of this approach is to force distressed companies to reduce the riskiness of their asset allocation. As mentioned earlier, similar conflicts of interests arise between debt and equity holders of companies. However, in the literature therein, this problem is usually tackled not by a change in the riskiness of the asset allocation but by a suitable design of capital structure (see, e.g., Leland [1996], Leland and Toft [1996]).

We first analyze the effect of a default constraint on the optimal asset allocation and assess whether this helps to (at least partially) solve the conflict of interests regarding the investment decision between insurance company and policyholders. Default is modeled continuously by a structural approach following the seminal paper of Black and Cox [1976]. In this first step, we assume that the insurance company commits to a constant-mix investment strategy at contract initiation and leaves this strategy unchanged until contract termination. In a second step, we then analyze a more flexible regulatory scheme: The regulator introduces a "traffic light system" that indicates whether the life insurance company is in danger of facing solvency problems ("yellow bulb") or even has severe and immediate problems ("red bulb"). If the insurance company is in distress ("yellow bulb"), the regulator may enforce a decrease in the riskiness of the asset allocation. This traffic light solvency stress test is for example implemented in Denmark and Sweden, see, e.g., Jørgensen [2007]. Similar ideas have been introduced in other European countries and in the Solvency II regulations. One

[2]Here, some authors suggest that the riskiness of the asset investment should depend on the insurance company's funding ratio (=assets divided by liabilities). An insurance company might adapt its asset allocation dependent on the possibility that it is (un)able to meet its obligations. Bohnert et al. [2015] suggest a CPPI-based strategy. Graf et al. [2011] and Hieber et al. [2015] change the asset allocation dependent on risk measures, i.e. the expected shortfall below the company's investment guarantees. Empirically, it is not obvious whether life insurance companies increase or decrease risk in case of distress: Mohan and Zhang [2014] find that US public funds increase risk if they are underfunded, while Rauh [2009] shows that the asset allocation is less risky if the company's financial condition is weaker.

advantage of the traffic light system is the fact that it is easy to implement and supervise. Furthermore – from a mathematical perspective – the contingent-claim framework is still analytically tractable, even in a continuous-time default setting. We investigate the effect of the traffic light system on the benefits of both the policyholders and the insurance company. If the regulator gets the possibility to force distressed insurance companies to decrease the riskiness of their investment strategy, this allows to significantly decrease solvency risk, only marginally changing the benefits of policyholders or the insurance company. If regulatory default constraints are the same under both the standard and the flexible regulatory framework, we show that the regulator might increase the benefits of policyholders and insurance company.

The remainder of the paper is organized as follows. In Section 2, we describe the model setup and introduce the payoffs of the policy- and shareholder. We set up their optimal investment problem, taking account of the possible default of the insurance company. More importantly, the flexible regulatory intervention (traffic light system) is presented. In the subsequent Section 3, the expected utility of the policy- and shareholder are computed analytically. We consider fairly-priced insurance contracts only and provide no-arbitrage conditions. In Section 4, we illustrate the advantage of the traffic light system in a numerical example. Therefore, we show that the traffic light system can increase the benefits of policy and shareholder (Pareto improvement). This effect is even more pronounced if the regulator enforces a default constraint. Finally, we provide some concluding remarks and an outlook for future research in Section 5 and detailed proofs in Section 6.

2. Notations and Model Setup

Our model contains three parties: an insurance regulator, a representative shareholder (also equity holder) and a representative policyholder (also liability holder). The latter two constitute a mutual life insurance company. We assume that the representative policyholder invests in a participating life insurance contract with a maturity of T years, $T < \infty$. At the initiation of the contract, the policyholder invests a lump sum L_0 in a single premium contract; the shareholder provides initial equity $E_0 > 0$. Consequently, the initial asset value A_0 of the insurance company is given by the sum of both contributions, i.e. $A_0 := L_0 + E_0$. We denote the share of the policyholder's contribution (or equivalently the debt ratio of our insurance company) by $\alpha := L_0/A_0$, where obviously $\alpha \in (0,1)$.

Asset model and guaranteed amount. Let us define a financial market consisting of one risk-free bond B with risk-free interest rate r, i.e. $\mathrm{d}B_t = rB_t\,\mathrm{d}t$ and $B_0 = 1$. Furthermore, there is the possibility to invest in a risky investment

$$\mathrm{d}S_t = \mu\, S_t\,\mathrm{d}t + \sigma\, S_t\,\mathrm{d}W_t, \quad S_0 = 1, \tag{1}$$

where $\mu > r$, $\sigma > 0$, and W is a standard Brownian motion under the real world measure $\mathbb{P}$. To start with, we assume that the insurance company invests the total proceeds A_0 in a diversified portfolio of risky and non-risky assets. Assume, a constant share $\theta_1 \in [0,1]$ is invested in the risky asset S and the remainder in the risk-free asset B. With the initial asset investment $A_0 > 0$, this yields the following asset dynamics:

$$\mathrm{d}A_t = \big(r + \theta_1(\mu - r)\big)\, A_t\,\mathrm{d}t + \sigma\theta_1\, A_t\,\mathrm{d}W_t\,. \tag{2}$$

The asset dynamic remains a log-normal process with a volatility of $\sigma\theta_1$. The amount guaranteed to the policyholder at time $t \in [0,T]$ is assumed to be $L_t = L_0\, e^{gt}$, where $g \leq r$ is the guaranteed rate. We want to draw conclusions regarding the financial risks of participating insurance contracts, therefore, as is usual in this context, we purely consider financial risks and ignore mortality risk (assume, for instance, that the fair market value of the contract is paid out immediately in case of death).

Default of the insurance company. We want to explicitly take the default risk of the insurance company into account. Therefore, we make use of a structural approach and assume that the insurance company defaults as soon as its assets A_t hit or drop below a specified percentage η of the guaranteed amount L_t. Thus, we introduce a default barrier $D_t := \eta L_0\, e^{gt}$ whose accrual rate g is the same as for the guaranteed amount. The time of default is then the first hitting time τ defined by

$$\tau := \inf\left\{t \geq 0 \,\middle|\, A_t \leq D_t\right\}, \tag{3}$$

where we set $\inf\{\emptyset\} = \infty$. The default parameter η is assumed to be smaller than A_0/L_0 such that the company is solvent initially.

Terminal payoff to liability and equity holder. The insurance payoff to the policyholder is contingent on whether the insurance company survives the maturity date T. If there is no premature default of the insurance company, the policyholder receives the following terminal payoff:

$$\Psi_L(A_T) := \begin{cases} A_T & \text{if } A_T \leq L_T \\ L_T + \delta\left[\alpha A_T - L_T\right]^+ & \text{else}, \end{cases}$$

$$= L_T + \delta\left[\alpha A_T - L_T\right]^+ - [L_T - A_T]^+ , \tag{4}$$

where we denote by $[\,\cdot\,]^+$ the maximum $\max\{\,\cdot\,, 0\}$. The participation rate $\delta \in [0, 1]$ is the percentage of surpluses that is credited to the liability holder. If there is no premature default, the terminal contract payoff is a combination of a fixed payment L_T, a bonus call and a shorted put option on the insurance company's assets. The shorted put option refers to losses of the liability holder if the company is not defaulted prematurely but assets at maturity are insufficient to cover the guaranteed amount.

In the case of premature default, a rebate payment is provided to the policyholder at time τ. This rebate payment is given by the minimum of the current asset value $A_\tau = D_\tau$ and the current liabilities L_τ: $\min(L_\tau, D_\tau)$. If we – for time consistency reasons – assume that the rebate payment is until T accumulated at the risk-free rate r, the policyholder receives the following contract payoff at time T:

$$V_L(A_T) := \mathbb{1}_{\{\tau > T\}}\, \Psi_L(A_T) + \mathbb{1}_{\{\tau \leq T\}}\, e^{r(T-\tau)} \min(L_\tau, D_\tau) , \tag{5}$$

where $\mathbb{1}_B$ is an indicator function which gives 1 if B occurs and 0 otherwise. The equity holder always obtains the residual asset value. If there is no premature default of the insurance company, the payoff to the equity holder is

$$\Psi_E(A_T) := \begin{cases} 0 & \text{if } A_T \leq L_T \\ A_T - L_T & \text{if } L_T < A_T \leq A_0 e^{gT} \\ A_T - L_T - \delta\left[\alpha A_T - L_T\right]^+ & \text{else} \end{cases}$$

$$= [A_T - L_T]^+ - \delta\left[\alpha A_T - L_T\right]^+ . \tag{6}$$

If there is premature default, a rebate payoff $D_\tau - \min(L_\tau, D_\tau)$ is provided to the equity holder. More compactly, the total payoff of the equity holder at maturity T is thus given by

$$V_E(A_T) := \mathbb{1}_{\{\tau > T\}}\, \Psi_E(A_T) + \mathbb{1}_{\{\tau \leq T\}}\, e^{r(T-\tau)} \max(D_\tau - L_\tau, 0) . \tag{7}$$

Hereby we have accrued the rebate payment at τ with the risk-free rate until the maturity date.

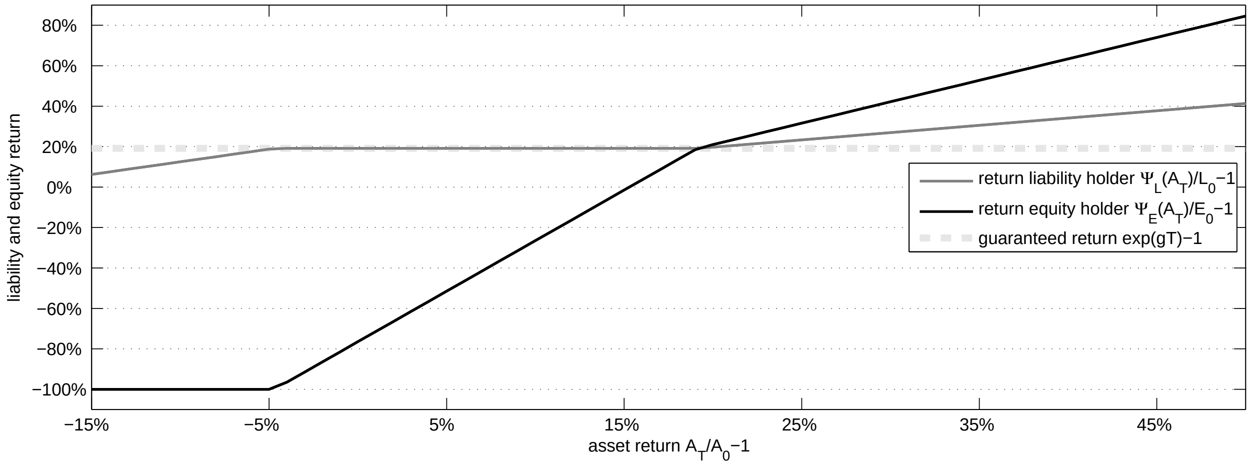

FIGURE 1. Return of liability and equity holder dependent on the asset return $A_T/A_0 - 1$ in case that the insurance company survives until maturity T, i.e. $\tau > T$. The parameters are set as $A_0 = 1$, $L_0 = \alpha A_0 = 0.8$, $D_0 = 0.85$, $\delta = 0.72$, $g = 1.75\,\%$, $T = 10$, $\mu = 6\,\%$, $r = 2.5\,\%$, $\theta_1 = 0.2$, and $\sigma = 0.2$.

If the company survives maturity T, the payoff to the liability and equityholder are given in Equation (4), respectively (6). A numerical example, presenting the return of liability and equity holder dependent on the asset return $A_T/A_0 - 1$, is given in Figure 1. The dashed line corresponds to the guaranteed return $\exp(gT) - 1$. If the asset return is greater than the guaranteed return, the liability holder (grey line) receives a bonus payment. If the insurance company survives the maturity, but the assets at that time are insufficient to cover the liabilities, the liability holder receives a return less than the guaranteed one. From Figure 1 one can observe that the equity holder's return (black line) is much more volatile than the liability holder's return.

Risk-neutrality of the equity holder. We assume that the insurance company has possibilities to diversify its investments and thus wants to maximize its expected payoff at maturity T. Its investment at contract initiation is given by $E_0 = (1 - \alpha)A_0$. The insurance company judges insurance contracts using the goal function

$$\mathcal{U}_E(\Theta) := \mathbb{E}_{\mathbb{P}}\left[V_E(A_T)\right], \tag{8}$$

where the parameter Θ contains the asset alloction parameters.[3]

[3]In case of the constant-mix investment strategy (2), we have that $\Theta = \theta_1$. Later on, the asset allocation strategy is allowed to be more flexible and, e.g., $\Theta = (\theta_1, \theta_2, K)$.

Liability holder. The liability holder – in contrast – cannot fully diversify the investment and is assumed to be risk-averse. She optimizes with respect to a utility function $u_L(\,\cdot\,)$ (see Definition 2.1) and thus evaluates her payments according to the goal function

$$\mathcal{U}_L(\Theta) := \mathbb{E}_{\mathbb{P}}\big[u_L\big(V_L(A_T)\big)\big]\,. \tag{9}$$

Definition 2.1 (Utility function). $u_L(\,\cdot\,)$ is increasing, concave, and twice differentiable on $\mathbb{R}$ with $u_L' > 0$, $u_L'' < 0$ $\lim_{x\to\infty} u_L(x) = -\infty$ and $\lim_{x\to\infty} u_L'(x) = \infty$.

Later on, we are exemplarily using power utility, see Example 2.2.

Example 2.2 (Power utility). *Let $\gamma_1 > 0$, $\gamma_1 \neq 1$ be the relative risk aversion parameter of the policyholder, i.e. $u_L(V_L) := V_L^{1-\gamma_1}/(1-\gamma_1)$.*

Competitive market. In our setting, we assume that the market is competitive and insurance contracts are fairly priced (see, e.g., Brennan and Schwartz [1976]). Policyholder are fully informed about available contract terms from different insurers. Arbitrage-free prices of insurance contracts are obtained under the pricing measure $\mathbb{Q}$ with the risk free bond $\mathrm{d}B_t/B_t = r\,\mathrm{d}t$ as reference asset. The risky asset evolves as

$$\mathrm{d}S_t = r\,S_t\,\mathrm{d}t + \sigma\,S_t\,\mathrm{d}W_t^{\mathbb{Q}}\,, \tag{10}$$

where still $B_0 = S_0 = 1$ and $W^{\mathbb{Q}}$ is a standard Brownian motion under $\mathbb{Q}$. If an insurance contract is fairly priced, the initial investment of the shareholder equals its arbitrage-free initial stake $E_0 = (1-\alpha)A_0$, i.e.

$$(1-\alpha)A_0 = \mathbb{E}_{\mathbb{Q}}\big[e^{-rT}\,V_E(A_T)\big]\,. \tag{11}$$

Similarly (and equivalently), from the policyholder's viewpoint, one has to ensure that

$$\alpha A_0 = \mathbb{E}_{\mathbb{Q}}\big[e^{-rT}\,V_L(A_T)\big], \tag{12}$$

with $V_E(A_T)$ and $V_L(A_T)$ as defined in Equation (7), respectively (5). We denote the set of fair contract terms Θ by $\mathcal{X}$.

Regulatory intervention. The aim of this paper is to examine the impact of regulation on the optimal asset allocation determined by the insurance company (see the optimization problem (8)). The regulator may force the insurance company to limit its default probability $\mathbb{P}(\tau \leq T)$ to an upper limit ϵ. If the reference portfolio has dynamics (2), we will later prove that this default constraint is equivalent to a bound on the share of risky assets θ_1, i.e. $0 \leq \theta_1 \leq \theta^{\max}$ for a given constant $\theta^{\max} \leq 1$. Both policy and equity holder value their contract terms using (9), respectively (8).

In a second step, the regulator allows for more flexibility in the investment strategy, while still keeping the default probability constraint $\mathbb{P}(\tau \leq T) \leq \epsilon$. The concept is in analogy to Solvency II regulations in Europe where the regulator has the possibility to intervene, as soon as the assets drop below some critical level $\{K_t\}_{t\geq 0}$ ("yellow bulb") to avoid a default event. If the company's assets nevertheless drop below the default barrier $\{D_t\}_{t\geq 0}$ ("red bulb"), the insurance company defaults. The possible interaction in case of the "yellow bulb" gives more freedom to act in the interests of both liability and equity holder. The second (upper) threshold K is set as

$$K_t := K_0 \, e^{gt} \,, \tag{13}$$

where $D_0 = \eta L_0 < K_0 < A_0$. The hitting time of this barrier is denoted

$$\hat{\tau} := \inf \left\{ t \geq 0 \,\middle|\, A_t \leq K_t \right\} , \tag{14}$$

where we again set $\inf\{\emptyset\} = \infty$. In case this barrier is hit, the regulator may once force the insurance company to change its investment strategy from θ_1 to $\theta_2 \in [0, 1]$. Then, the asset value process is – for $t \geq 0$ – given by

$$\mathrm{d}A_t = \big(r + \theta_{Z_t}(\mu - r)\big)\, A_t \,\mathrm{d}t + \theta_{Z_t}\sigma \, A_t \,\mathrm{d}W_t, \quad A_0 > 0, \tag{15}$$

where $Z_t = 1$ for $t \leq \hat{\tau}$ and $Z_t = 2$ for $t > \hat{\tau}$. The effect of this more flexible design on the benefits of equity and liability holder is analyzed in the remainder of this paper. For reasons of analytical tractability, we do not consider a strategy recovery of the insurance company, i.e. it is not possible to return to the original asset strategy θ_1.

Under this more flexible regulation, the default-triggering event remains unchanged. A default occurs when the asset process A_t hits the lower threshold D_t (i.e. if $\{\tau \leq T\}$). Since the asset process is continuous and the regulatory barrier K_t by definition greater than D_t, the event $\{\tau \leq T\}$ implies that $\{\hat{\tau} \leq T\}$, i.e. the upper threshold is hit before the lower one.[4]

[4]The event $\{\hat{\tau} > T\}$ delineates the situation that the assets perform well until maturity T and all the time exceed the upper regulatory threshold. The event $\{\hat{\tau} \leq T, \tau > T\}$ describes the situation that the assets perform moderately until maturity T. The assets have hit the regulatory barrier but the insurance company has not defaulted prematurely. The event $\{\tau \leq T\}$ describes the situation that the company has defaulted.

3. Theoretical results

In order to determine the optimal investment strategy and examine the regulatory effects on it, we need to compute the expected payoff of the equity holder (8) and the expected utility of the policyholder (9).

3.A. **Traditional regulatory framework.** In this first case, we assume that there is no regulatory barrier $\{K_t\}_{t\geq 0}$ and thus the investment strategy stays constant at $\theta_1 \in [0,1]$. Theorem 3.1 gives analytical expressions for the expected payoff to the equity holder and the expected utility of terminal payoffs to the policyholder.

Theorem 3.1 (Expected utility: No intervention). *Assume the model setup as described in Section 2 with asset process (2). Then, the desired expectations are given by*

$$\mathcal{U}_L(\Theta) =: \kappa_L^{(1)}(A_0, D_0, L_0, 0, T), \quad \mathcal{U}_E(\Theta) =: \kappa_E^{(1)}(A_0, D_0, L_0, 0, T)\,,$$

where $\kappa_L^{(1)}(\,\cdot\,)$ *and* $\kappa_E^{(1)}(\,\cdot\,)$ *can be computed via*

$$\begin{aligned}\kappa_L^{(i)}(A_t, D_t, L_t, t, T) &= \int_t^T u_L\Big(e^{r(T-t)+(g-r)(\tau-t)}\min(L_t, D_t)\Big)\, f^{(i)}(t,\tau,A_t,D_t)\,d\tau \\ &+ \int_{\ln(D_t/A_t)}^{\infty} u_L\big(L_T + \delta\big[\alpha A_t e^{y+g(T-t)} - L_T\big]^+ - \big[L_T - A_t e^{y+g(T-t)}\big]^+\big)\, g^{(i)}(y,t,T,A_t,D_t)\,dy\,,\end{aligned}$$

$$\begin{aligned}\kappa_E^{(i)}(A_t, D_t, L_t, t, T) &= \int_t^T e^{r(T-t)+(g-r)(\tau-t)}\max(D_t - L_t, 0)\, f^{(i)}(t,\tau,A_t,D_t)\,d\tau \\ &+ \int_{\ln(D_t/A_t)}^{\infty} \big(\big[A_t e^{y+g(T-t)} - L_T\big]^+ - \delta\big[\alpha A_t e^{y+g(T-t)} - L_T\big]^+\big)\, g^{(i)}(y,t,T,A_t,D_t)\,dy\,.\end{aligned}$$

The densities g*, respectively* f*, are defined as*

$$g^{(i)}(y,t,T,A_t,D_t) := \frac{1}{\sigma\theta_i\sqrt{T-t}}\,\varphi\left(\frac{y-\tilde{\mu}_i(T-t)}{\sigma\theta_i\sqrt{T-t}}\right)\left(1 - e^{-2\,\frac{\ln(D_t/A_t)^2 - y\ln(D_t/A_t)}{\sigma^2\theta_i^2(T-t)}}\right),$$

$$f^{(i)}(t,\tau,A_t,D_t) := \frac{-\ln(D_t/A_t)}{\sigma\theta_i(\tau-t)^{\frac{3}{2}}}\,\varphi\left(\frac{\ln(D_t/A_t) - \tilde{\mu}_i(\tau-t)}{\sigma\theta_i\sqrt{\tau-t}}\right),\ \tilde{\mu}_i := r + \theta_i(\mu - r) - g - \sigma^2\theta_i^2/2\,,$$

where $\varphi(\,\cdot\,)$ *denotes the density of the standard normal distribution.*

Proof: See the Appendix.

In the case of power utilities, most of the integrals in Theorem 3.1 can be derived analytically. The default probability on the time interval $[0, T]$ is given by

$$\mathbb{P}(\tau \leq T) = \Phi\left(\frac{\ln(D_0/A_0) - \tilde{\mu}_1 T}{\sigma\theta_1\sqrt{T}}\right) + \left(\frac{D_0}{A_0}\right)^{\frac{2\tilde{\mu}_1}{\sigma^2\theta_1^2}} \Phi\left(\frac{\ln(D_0/A_0) + \tilde{\mu}_1 T}{\sigma\theta_1\sqrt{T}}\right), \tag{16}$$

where $\Phi(\,\cdot\,)$ is the standard normal distribution function and $\tilde{\mu_1}$ is defined as in Theorem 3.1, see also the Appendix. From Theorem 3.1, we can conclude that the fair pricing condition (11) is valid if and only if we solve the equation $\kappa_E^{(1)}(A_0, D_0, L_0, 0, T) = (1-\alpha)A_0 e^{rT}$ for δ.

3.B. **Flexible regulatory framework.** Now, we are going to derive the same results as in Theorem 3.1 under the assumption that the investment strategy is changed from θ_1 to θ_2 as soon as the regulatory barrier $\{K_t\}_{t\geq 0}$ is hit. This leads to the asset process given by (15). Technically, this setup is still analytically tractable: Until first hitting the regulatory threshold K at time $\hat{\tau}$, the asset process behaves as a geometric Brownian motion – one of the rare cases where the first-hitting time density is known analytically (the hitting time is distributed according to an inverse Gaussian law, see, for example, Folks and Chhikara [1978]). At time $\hat{\tau}$, the assets equal the barrier $K_{\hat{\tau}}$. After this hitting time, the assets are again a geometric Brownian motion now with a different mean and volatility parameter due to the changed investment strategy θ_2. Thus, the time to default follows again an inverse Gaussian law. To sum up, the default time τ is given by the convolution of two inverse Gaussian random variables. The default probability can be evaluated via

$$\begin{aligned}\mathbb{P}(\tau \leq T) &= \int_0^T \mathbb{P}\big(\tau \leq T \,\big|\, A_{\hat{\tau}} = K_{\hat{\tau}}\big) \cdot f^{(1)}(0, \hat{\tau}, A_0, K_0)\,\mathrm{d}\hat{\tau} \\ &= \int_0^T \int_0^{T-\tau} f^{(2)}(\hat{\tau}, \tau, K_{\hat{\tau}}, D_{\hat{\tau}}) \cdot f^{(1)}(0, \hat{\tau}, A_0, K_0)\,\mathrm{d}\hat{\tau}\,\mathrm{d}\tau\,,\end{aligned} \tag{17}$$

with f as defined in Theorem 3.1. Equation (16) results as the special case $\theta_1 = \theta_2$. Similarly to Theorem 3.1, one can derive the expected terminal payoff to the equity holder $\mathbb{E}_{\mathbb{P}}[V_E(A_T)]$ and the expected utility $\mathbb{E}_{\mathbb{P}}[u_L(V_L(A_T))]$ of the liability holder, see Theorem 3.2.

THEOREM 3.2 (Expected utility: Regulatory intervention). *Assume the model setup as described in Section 2 with asset process (15). The regulator may intervene at time $\hat{\tau}$ – the first hitting time of the insurance company's assets A breaching the regulatory barrier $K_t = K_0 e^{gt}$. At time $\hat{\tau}$, the insurance company is forced to change its investment strategy from θ_1 to θ_2.*

Then, the desired expectations are given by

$$\mathcal{U}_L(\Theta) =: \zeta_L(A_0, D_0, K_0, L_0, 0, T), \quad \mathcal{U}_E(\Theta) =: \zeta_E(A_0, D_0, K_0, L_0, 0, T)\,,$$

where

$$\begin{aligned}
\zeta_L(A_t, D_t, K_t, L_t, t, T) = & \int_t^T \int_{\ln(D_t/K_t)}^{\infty} u_L\big(L_T + \delta\big[\alpha K_{\hat\tau} e^{y+g(T-\hat\tau)} - L_T\big]^+ - \big[L_T - K_{\hat\tau} e^{y+g(T-\hat\tau)}\big]^+\big) \\
& \cdot f^{(1)}(t, \hat\tau, A_t, K_t) \cdot g^{(2)}(y, \hat\tau, T, K_{\hat\tau}, D_{\hat\tau})\; dy\, d\hat\tau \\
& + \int_t^T \int_{\hat\tau}^T u_L\Big(e^{r(T-t)+(g-r)(\hat\tau-t)} \min(L_t, D_t)\Big) \cdot f^{(1)}(t, \hat\tau, A_t, K_t) \cdot f^{(2)}(\hat\tau, \tau, K_{\hat\tau}, D_{\hat\tau})\; d\tau\, d\hat\tau \\
& + \int_{\ln(K_t/A_t)}^{\infty} u_L\big(L_T + \delta\big[\alpha A_t e^{y+g(T-t)} - L_T\big]^+ - \big[L_T - A_t e^{y+g(T-t)}\big]^+\big)\, g^{(1)}(y, t, T, A_t, K_t)\, dy
\end{aligned}$$

$$\begin{aligned}
\zeta_E(A_t, D_t, K_t, L_t, t, T) = & \int_t^T \int_{\ln(D_t/K_t)}^{\infty} \big(\big[K_{\hat\tau} e^{y+g(T-\hat\tau)} - L_T\big]^+ - \delta\big[\alpha K_{\hat\tau} e^{y+g(T-\hat\tau)} - L_T\big]^+\big) \\
& \cdot f^{(1)}(t, \hat\tau, A_t, K_t) \cdot g^{(2)}(y, \hat\tau, T, K_{\hat\tau}, D_{\hat\tau})\; dy\, d\hat\tau \\
& + \int_t^T \int_{\hat\tau}^T e^{r(T-t)+(g-r)(\hat\tau-t)} \max(D_t - L_t, 0)\, f^{(1)}(t, \hat\tau, A_t, K_t) \cdot f^{(2)}(\hat\tau, \tau, K_{\hat\tau}, D_{\hat\tau})\; d\tau\, d\hat\tau \\
& + \int_{\ln(K_t/A_t)}^{\infty} \big(\big[A_t e^{y+g(T-t)} - L_T\big]^+ - \delta\big[\alpha A_t e^{y+g(T-t)} - L_T\big]^+\big)\, g^{(1)}(y, t, T, A_t, K_t)\, dy\,,
\end{aligned}$$

with f and g as defined in Theorem 3.1.

Proof: See the Appendix.

Again, it is straightforward to rephrase the fair pricing condition (11) in terms of the participation rate δ, i.e. the insurance contract is fairly priced if and only if we solve the equation $\zeta_E(A_0, D_0, K_0, L_0, 0, T) = (1-\alpha)A_0 e^{rT}$ for δ.

Remark 3.3 (Implementation of Theorems 3.1 and 3.2). The expectations presented in Theorems 3.1 and 3.2 are integrals over normal densities. Thus, they can easily be implemented at high precision. Computation time is within fractions of seconds.

That is why it does not make sense to further simplify the given expressions and solve the integrals analytically, although it is, for example, possible to present $\kappa_E^{(i)}(A_t, D_t, L_t, t, T)$ in Theorem 3.1 in a (lengthy) closed-form expression.

3.C. **Asset allocation decision.** So far, we have shown how to compute the goal function pairs $(\mathcal{U}_L(\Theta), \mathcal{U}_E(\Theta))$ of policyholder and equity holder. The goal function pairs are determined under the assumption of a competititve market, i.e. we have shown how the participation rate can be set in order to fulfill the fair pricing condition (11). In the remainder of the paper we want to focus on the following two questions:

(1) Is there some optimality criterium for the insurance contracts in our model setup? Can we choose the asset allocation parameters $\Theta = (\theta_1, \theta_2, K_0)$ in order to "maximize" the pair $(\mathcal{U}_L(\Theta), \mathcal{U}_E(\Theta))$?
(2) How does a regulatory default constraint affect the considerations in (1)?

A general answer to (1) is impossible as this requires to find a reasonable way to somehow weigh the goals of liability and equity holder. However, there should be broad consensus that if a given contract with utility pair $(\mathcal{U}_L(\Theta^*), \mathcal{U}_E(\Theta^*))$ strictly dominates another pair $(\mathcal{U}_L(\Theta), \mathcal{U}_E(\Theta))$, i.e. if $\mathcal{U}_L(\Theta^*) > \mathcal{U}_L(\Theta)$ and $\mathcal{U}_E(\Theta^*) > \mathcal{U}_E(\Theta)$, then this second contract is in some sense sub-optimal. That is why, following, e.g., Bokrantz and Fredriksson [2014], Filipović et al. [2015], we define Pareto-efficient contracts via Definition 3.4.

Definition 3.4 (Pareto-efficient contract terms). Contract terms $\Theta^* = (\theta_1, \theta_2, K_0) \in \mathcal{X}$ are Pareto-efficient if there does not exist a contract term $\Theta \in \mathcal{X}$ such that

$$\Theta^* \succ \Theta,$$

meaning that at least one of the following holds:

$$\mathcal{U}_L(\Theta^*) \geq \mathcal{U}_L(\Theta) \text{ and } \mathcal{U}_E(\Theta^*) > \mathcal{U}_E(\Theta), \quad \text{or} \quad \mathcal{U}_L(\Theta^*) > \mathcal{U}_L(\Theta) \text{ and } \mathcal{U}_E(\Theta^*) \geq \mathcal{U}_E(\Theta).$$

If the set of feasible contract terms $\mathcal{X}$ (i.e. the contract terms that fulfill the fair pricing condition (11)) is non-empty and compact, Theorem 3.5 gives conditions for the existence of Pareto-efficient contract terms.

Theorem 3.5 (Conditions for Pareto-efficiency). *Assume that the set of feasible contract terms $\mathcal{X}$ is non-empty and compact and that both $\mathcal{U}_L(\Theta)$ and $\mathcal{U}_E(\Theta)$ are lower semicontinuous in Θ. Then, there exists a set of Pareto-efficient contracts.*

Proof: See, e.g., Bokrantz and Fredriksson [2014] and the references therein.

We can apply Theorem 3.5 to our setting. Therefore, first define the set of feasible contract terms by contracts that fulfill the fair pricing condition (11):

$$\mathcal{X} := \left\{ \Theta \,\middle|\, (1-\alpha)A_0 = \mathbb{E}_{\mathbb{Q}}\left[e^{-rT}\, V_E(A_T)\right] \right\}, \tag{18}$$

where the distribution of A_T depends on the asset allocation parameters $\Theta = (\theta_1, \theta_2, K_0)$. As $0 \leq \theta_1, \theta_2 \leq 1$ and $D_0 \leq K_0 \leq A_0$, this set is obviously compact. Further, as we can see from the smooth integrands in Theorem 3.2, the functions $\mathcal{U}_L(\Theta)$ and $\mathcal{U}_E(\Theta)$ are continuous in Θ. Thus, if we assume that there exists at least one fairly priced contract, then, according to Theorem 3.5, there exists a set of Pareto-efficient contracts.

If a regulator is introduced, the set of feasible contracts additionally has to fulfill the default constraint $\mathbb{P}(\tau \leq T) \leq \epsilon$. Then

$$\tilde{\mathcal{X}} := \left\{ \Theta \,\middle|\, (1-\alpha)A_0 = \mathbb{E}_{\mathbb{Q}}\left[e^{-rT}\, V_E(A_T)\right];\, \mathbb{P}(\tau \leq T) \leq \epsilon \right\}. \tag{19}$$

As $\mathbb{P}(\tau \leq T)$ is continuous in the asset allocation parameters, the compactness of $\mathcal{X}$ implies the compactness of $\tilde{\mathcal{X}}$. Thus, again, we conclude from Theorem 3.5: If there is a fair contract that fulfills the default constraint, then there exists a set of Pareto-efficient contracts.

4. Numerical illustrations

We now illustrate the notion of Pareto-efficiency in a numerical example. We show that the flexible regulatory scheme can be beneficial to both the liability and the equity holder (Pareto improvement). Further, due to the non-linearity of the contract payoff, we show that it is not the Merton-type constant-mix strategy that maximizes the liability holder's utility.

Therefore, we first fix the contract parameters that are the same in both the traditional and the flexible regulatory framework. In contrast to the asset allocation strategy, those parameters are either set or restricted by national law (insolvency condition η) or changing them might be difficult for the insurance company (equity share $1-\alpha$):

- In line with balance sheet data from big German insurance companies where equity is mainly composed of stocks and undistributed reserves, we set the equity share to $1-\alpha = 0.1$. Form this, the initial assets are (without loss of generality) $A_0 = 1$; the initial single premium is $L_0 = 0.9$.

- We assume that the insurance company defaults as soon as the assets drop below the guaranteed amount L_t, i.e. $D_0 = L_0 = 0.9$, $\eta = 1$, an assumption that is common in the context of structural models (see, e.g., Brennan and Schwartz [1976]).
- The utility function of the policyholder is assumed to be power utility with a relative risk aversion parameter $\gamma_1 = 3$.

The participation rate δ is set according to the fair pricing condition (11). Note that in some countries, legal constraints on δ are imposed. If this is the case, we can incorporate this fact in our analysis by including this constraint in our definition of feasible contract terms $\mathcal{X}$. Further, we set the time horizon to $T = 10$, and adapt the financial market parameters to the current situation in Germany: $\mu = 6\,\%$, $r = 2.5\,\%$, and $\sigma = 0.2$. The guaranteed rate is set to 1.25%, which corresponds to the current maximum technical interest rate in Germany.

The qualitative results in this section are consistent if the parameterd $\alpha, \eta, \gamma_1, T, \mu, r, \sigma$ are changed in a reasonable way (e.g. if an underfunding of the insurance company is allowed by law, and thus $\eta < 1$).

In the following, we want to optimally determine the asset allocation parameters in case of the traditional ($\Theta = (\theta_1, \theta_1, \cdot)$) and the flexible regulatory scheme ($\Theta = (\theta_1, \theta_2, K_0)$). Therefore, we first consider the case without default constraint by the regulator (feasible set $\mathcal{X}$); then we assume that this default constraint is imposed (feasible set $\tilde{\mathcal{X}}$).

Traditional regulatory framework

No.	strategy θ_1	$(\mathcal{U}_L(\Theta), \mathcal{U}_E(\Theta))$	PD
1a	$\theta_1 = 0.180$	$(-0.3486, 0.1512)$	0.46%
2a	$\theta_1 = 1.000$	$(-0.3669, 0.3010)$	14.77%
3a	$\theta_1 = 0.183$	$(-0.3486, 0.1521)$	0.50%

Table I. Different asset allocation strategies in a traditional regulatory framework (constant-mix, $\theta_1 = \theta_2$). For each strategy, we comupute the utility pair $(\mathcal{U}_L(\Theta), \mathcal{U}_E(\Theta))$ and an annualized default probability $PD := 1 - \left(1 - \mathbb{P}(\tau \leq T)\right)^{1/T}$.

Before we further examine the efficiency condition in Theorem 3.5, we want to look at the effect of the investment strategy (θ_1, θ_2) on the goal function pairs $(\mathcal{U}_L(\Theta), \mathcal{U}_E(\Theta))$. First, we deal with the traditional (constant-mix) regulatory framework. Table I presents the optimal

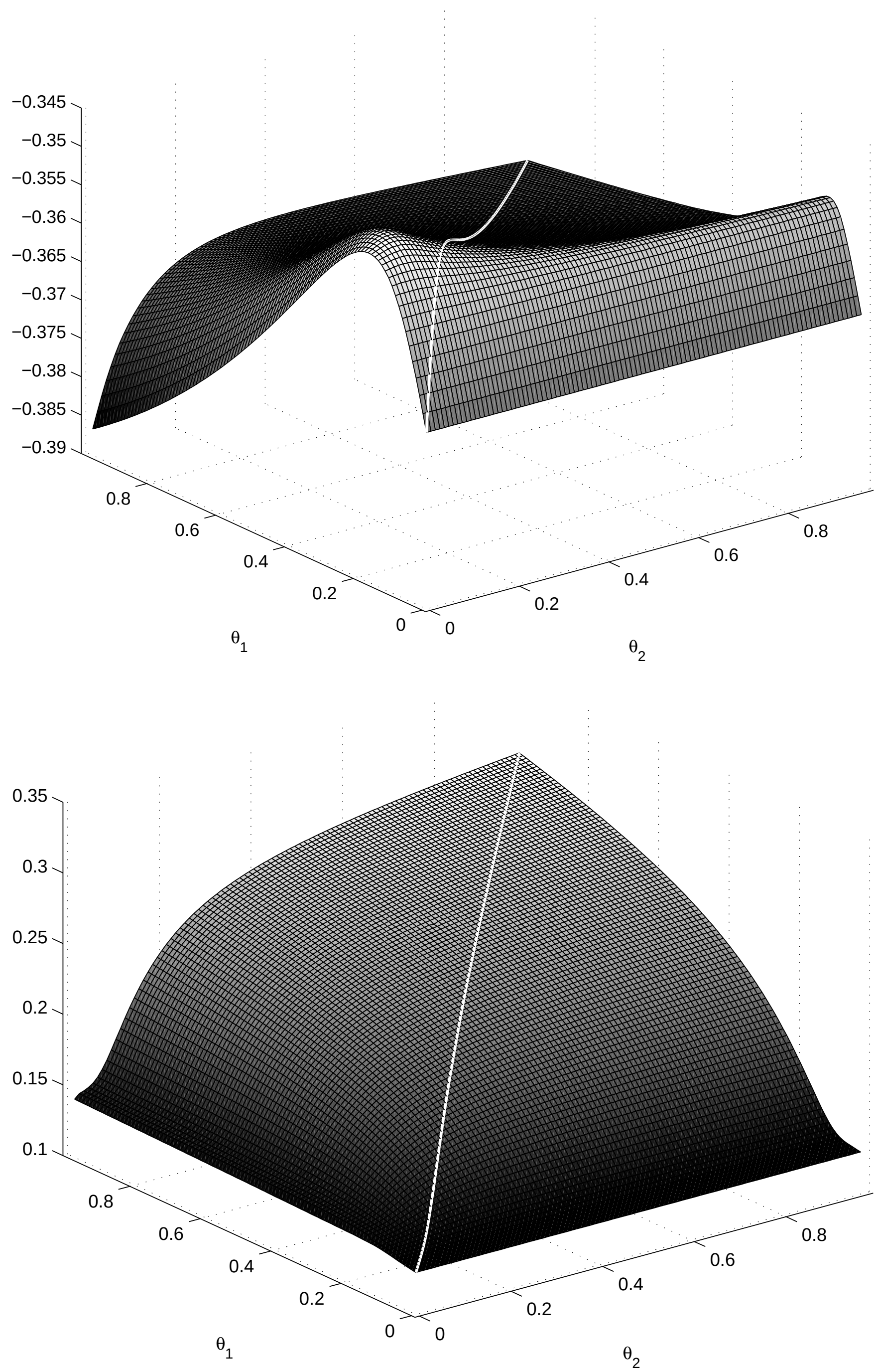

FIGURE 2. Goal function of the liability holder $\mathcal{U}_L(\Theta)$ (top) and equity holder $\mathcal{U}_E(\Theta)$ (below) in a numerical example dependent on the investment strategy (θ_1, θ_2). We set $K_0 = 0.94$, $A_0 = 1$, $L_0 = D_0 = 0.9$, $g = 1.25\,\%$, $T = 10$, $\mu = 6\,\%$, $r = 2.5\,\%$, and $\sigma = 0.2$. The contracts are initially fairly priced, i.e. $\Theta \in \mathcal{X}$.

asset allocation of the liability holder (strategy 1a) and the equity holder (strategy 2a). Furthermore, it presents the riskiest strategy that is in line with a Solvency II annualized default probability of 0.5%.[5] Note that, in our example, the optimal strategy 1a of the liability holder is a feasible strategy even if a default probability constraint is imposed ($PD \leq 0.50\%$). If this default constraint is further tightened or the parameter set changed, it might happen that both the liability and the equity holder's optimal asset allocation strategy leads to an annualized default probability PD that is greater than 0.50%. In this case, strategy 3a Pareto-dominates strategy 1a.

Next, we consider the flexible regulatory scheme. For illustratory purposes, we for the moment fix the regulatory threshold to $K_0 = 0.94$. Figure 2 presents the goal function pairs $\mathcal{U}_L(\Theta)$ (top) and $\mathcal{U}_E(\Theta)$ (below) for different asset allocation strategies (θ_1, θ_2). Therefore, we use a grid of length 0.01 for both variables $\theta_1 \in [0,1]$ and $\theta_2 \in [0,1]$. The white line corresponds to the case of no regulatory intervention, i.e. $\theta_1 = \theta_2$. We first observe that the equity holder (below) seems to prefer more risky asset allocations – his goal function is increasing in both θ_1 and θ_2. He obtains a global optimum for $\theta_1 = \theta_2 = 1$ – an optimum that is unchanged if the flexible regulatory scheme is implemented. The liability holder's preferences are different: If there is no regulatory intervention (white line), the maximum utility $\mathcal{U}_L(\Theta) = -0.3486$ is reached for $\theta_1 = \theta_2 = 0.18$. In the flexible regulatory scheme, however, the global optimum for the liability holderis a utility of $\mathcal{U}_L(\Theta) = -0.3458$ if $(\theta_1, \theta_2, K_0) = (0.22, 0.07, 0.94)$. This already indicates that the liability holder would prefer a scheme that decreases risk in distress (i.e. $\theta_2 < \theta_1$).

To return to our definition of Pareto efficiency in Theorem 3.5, it makes sense to no longer set $K_0 = 0.94$ but to additionally optimize over the parameter K_0. Then, we can present goal function pairs $(\mathcal{U}_L(\Theta), \mathcal{U}_E(\Theta))$ for each asset allocation strategy $(\theta_1, \theta_2, K_0)$, see Figure 3 (left hand side). Again, the case without any regulatory intervention is represented by the white line. Each Pareto-efficient portfolio, i.e. a portfolio that is not dominated by another utility pair $(\mathcal{U}_L(\Theta), \mathcal{U}_E(\Theta))$, lies on the upper edge of the black area. We observe that each

[5] In order to interpret the results, we anualize default probabilities to $PD := 1 - \big(1 - \mathbb{P}(\tau \leq T)\big)^{1/T}$. Note, however, that the structural model implies a term structure of default probabilities that cannot be summarized by just one number. The value presented can thus be seen as some average yearly default probability of the insurance company.

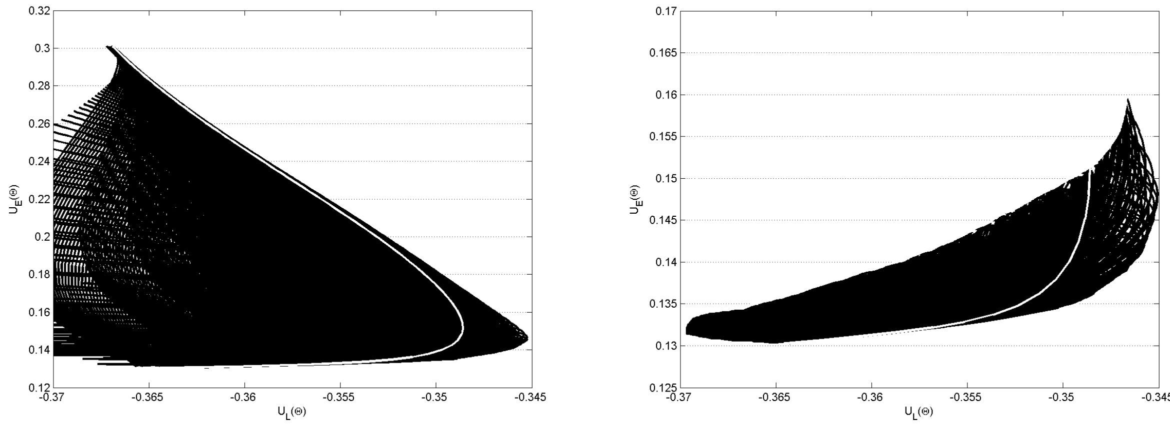

FIGURE 3. Goal function pairs $(\mathcal{U}_L(\Theta), \mathcal{U}_E(\Theta))$. Each point in this graph corresponds to one asset allocation strategy $(\theta_1, \theta_2, K_0)$. The left hand side presents all possible pairs; the right hand side only contracts whose annualized default probability $PD := 1 - \big(1 - \mathbb{P}(\tau \leq T)\big)^{1/T}$ is smaller than 0.5%. The constant-mix strategy $(\theta_1 = \theta_2)$ is given by the white line.

contract from the traditional regulatory framework (white line) is always Pareto-dominated by several contracts from the flexible regulatory framework (black area). The right hand side of Figure 3 presents the subset of contracts whose annualized default probability is lower than 0.5% (feasible set $\tilde{\mathcal{X}}$).

To demonstrate this fact more clearly, Table II takes some specific asset allocation strategies. For each strategy, the utility pair $(\mathcal{U}_L(\Theta), \mathcal{U}_E(\Theta))$ and annualized default probabilities are computed.[6] First, the table presents the optimal asset allocation strategy of the liability holder (strategy 1b) and the equity holder (strategy 2b) under the flexible regulatory scheme. As the equity holder prefers the highest risk possible, his results are unchanged, i.e. 2a and 2b coincide.[7] For the liability holder, however, results are different: His optimal utility

[6]In our example, the default probability is increasing in both θ_1 and θ_2. Intuitively, a higher equity holding (leading to a higher volatility) results in a higher probability of hitting the default barrier D_t. If there is no regulatory barrier K_t, this can easily be proved analytically, i.e. from Equation (16) we obtain

$$\frac{\partial\, \mathbb{P}(\tau \leq T)}{\partial \theta_1} = \frac{-2\ln(D_0/A_0)}{\sigma\theta_1^2\sqrt{T}}\,\varphi\!\left(\frac{\ln(D_0/A_0) - \tilde{\mu}_1 T}{\sigma\theta_1\sqrt{T}}\right) - \ln(D_0/A_0)\left(\frac{4(r-g)T}{\sigma^2\theta_1^3} + \frac{\mu - r}{\sigma^2\theta_1^2}\right)\left(\frac{D_0}{A_0}\right)^{\frac{2\tilde{\mu}_1}{\sigma^2\theta_1^2}}\,\Phi\!\left(\frac{\ln(D_0/A_0) + \tilde{\mu}_1 T}{\sigma\theta_1\sqrt{T}}\right) > 0\,,$$

since $D_0 < A_0$ and $g \leq r < \mu$.

[7]Note that for $\theta_1 = \theta_2$, the asset allocation strategy does not depend on the parameter K_0.

Flexible regulatory framework

No.	strategy $(\theta_1, \theta_2, K_0)$	$(\mathcal{U}_L(\Theta), \mathcal{U}_E(\Theta))$	PD
1b	$(0.230, 0.040, 0.91)$	$(-0.3451, 0.1489)$	0.00%
2b	$(1.000, 1.000, \ \cdot\ )$	$(-0.3669, 0.3010)$	14.77%
3b	$(0.240, 0.110, 0.92)$	$(-0.3468, 0.1581)$	0.50%

Table II. Different asset allocation strategy in a flexible regulatory framework (asset allocation is changed from θ_1 to θ_2 if regulatory barrier $K_t = K_0 e^{gt}$ is hit). For each strategy, the utility pair $(\mathcal{U}_L(\Theta), \mathcal{U}_E(\Theta))$ and an annualized default probability $PD := 1 - \big(1 - \mathbb{P}(\tau \leq T)\big)^{1/T}$ is computed.

is not obtained under a constant-mix investment strategy. Instead, he prefers the strategy $(\theta_1, \theta_2, K_0) = (0.23, 0.04, 0.91)$, a strategy with a slightly higher initial share in the risky asset and a stronger downside protection. Strategy 3b is Pareto-optimal if contracts have to fulfill the solvency requirement $PD \leq 0.5\%$ (in this case the set of feasible contracts is $\tilde{\mathcal{X}}$; the results in Theorem 3.5 still apply). Note that contract 3b dominates contract 3a from Table I.

To summarize our numerical illustrations, note that the flexible regulatory framework allows us to find fair contract terms that increase the goal function of both the liability holder and the equity holder (Pareto improvement). An asset allocation with initially (θ_1) higher equity ratios and a reduction in case of distress $(\theta_2 < \theta_1)$ is benefical for liability and equity holder.

5. Conclusion

The present paper discusses flexible regulatory supervision to partly solve the conflict of interests that arises by the option-like stakes of the insurance company. Typically, the regulator imposes a Value-at-Risk-type constraint (default probability constraint) on the investment strategy. In view of the option-like stakes and non-linear sharing rules of a participation life insurance contract, we show that the regulator should not only impose a default constraint. We show that an intervention of the regulator in case of distress forcing the insurance company to decrease risk in distress, might mitigate the conflict of interests between policy- and shareholder and lead to Pareto-improvements for the two involved parties.

The advantage of this flexible scheme is that it is rather simple and – in contrast to, for example, a more flexible and dynamic asset allocation strategy like CPPI – easy to implement and supervise for a regulatory authority.

At least from an academic perspective, it is, however, desirable to generalize the results to other dynamic asset allocation strategies. In our opinion, this is an important but challenging further research direction. It is at least possible to allow for more regulatory barriers to enlarge the set of possible investment strategies. An extension to general dynamic asset allocation strategy seems to be difficult – the problem lies in the non-linearity of the contract payoff and the fact that the asset allocation is optimized subject to a fair pricing constraint.

6. Appendix

Proof of Theorem 3.1. First, we recall results on the first-hitting time τ of a geometric Brownian motion, i.e. the process A as defined in (2). The law of τ is known to be inverse Gaussian (see, e.g., Folks and Chhikara [1978]). Lemma 6.1 recalls some results on the first-hitting time in this special case.

Lemma 6.1 (First-hitting time distribution). *Consider the process A from (2). Then, the survival probability within the interval $(t, T]$ is given by*

$$\mathbb{P}\big(\tau > T \,|\, \tau > t\big) = \Phi\left(\frac{\tilde{\mu}_1(T-t) - \ln(D_t/A_t)}{\sigma\theta_1\sqrt{T-t}}\right) - \left(\frac{D_t}{A_t}\right)^{\frac{2\tilde{\mu}_1}{\sigma^2\theta_1^2}} \Phi\left(\frac{\tilde{\mu}_1(T-t) + \ln(D_t/A_t)}{\sigma\theta_1\sqrt{T-t}}\right),$$

where $D_t < A_t$, $\tilde{\mu}_1 := r + \theta_1(\mu - r) - g - \sigma^2\theta_1^2/2$, and $\Phi(\,\cdot\,)$ denotes the standard normal cumulative distribution function. The density of τ can be obtained from

$$f^{(i)}(t, \tau, A_t, D_t) := \frac{-\ln(D_t/A_t)}{\sigma\theta_i(\tau - t)^{\frac{3}{2}}}\, \varphi\left(\frac{\ln(D_t/A_t) - \tilde{\mu}_i(\tau - t)}{\sigma\theta_i\sqrt{\tau - t}}\right). \tag{20}$$

For $y := \ln(e^{-gT} A_T/A_t)$, we define

$$g^{(1)}(y, t, T, A_t, D_t) := \mathbb{P}\big(y \in dy,\, \tau > T\big)\,, \tag{21}$$

which is known to be

$$g^{(1)}(y, t, T, A_t, D_t) = \begin{cases} 0 & \text{for } y \le \ln(D_t/A_t) \\ \frac{\varphi\left(\frac{y - \tilde{\mu}_i(T-t)}{\sigma\theta_1\sqrt{T-t}}\right)}{\sigma\theta_1\sqrt{T-t}} \left(1 - e^{-2\,\frac{\ln(D_t/A_t)^2 - y\ln(D_t/A_t)}{\sigma^2\theta_1^2(T-t)}}\right) & \text{else} \end{cases} \tag{22}$$

where $\varphi(\,\cdot\,)$ denotes the density of the standard normal distribution.

Proof: See, e.g., Folks and Chhikara [1978], He et al. [1998], and Shreve [2004].

Note that the same results hold, if we replace the asset strategy θ_1 by θ_2 (and similarly $\tilde{\mu}_1$ by $\tilde{\mu}_2 := r + \theta_2(\mu - r) - g - \sigma^2\theta_2^2/2$). We denote the densities that result from this parameter change by $f^{(2)}(t, \tau, A_\tau, D_\tau)$, respectively $g^{(2)}(y, t, T, A_t, D_t)$.

We are now using Lemma 6.1 to prove Theorem 3.1. Note first that if the barrier D is not hit in the interval $(t, T]$, (22) helps us to obtain the distribution of the assets A at maturity T. To compute the expected utility of the terminal payoff $\Psi_L(A_T)$ from (4), one simply has to integrate its utility over (22) on the set $(\ln(e^{-g(T-t)}\, D_T/A_t), \infty) = (\ln(D_t/A_t), \infty)$. If the barrier is hit, i.e. $\tau \le T$, the terminal payoff depends solely on the default time τ whose distribution can be obtained from (20). This then leads to

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}\big[u_L(V_L)\big] &= \mathbb{E}_{\mathbb{P}}\big[u_L\big(\mathbb{1}_{\{\tau>T\}}\,\Psi_L(A_T) + \mathbb{1}_{\{\tau\le T\}}\, e^{r(T-\tau)}\,\min(L_\tau, D_\tau)\big)\big] \\
&= \mathbb{E}_{\mathbb{P}}\big[\mathbb{1}_{\{\tau>T\}}\, u_L\big(\Psi_L(A_T)\big)\big] + \mathbb{E}_{\mathbb{P}}\big[\mathbb{1}_{\{\tau\le T\}}\, u_L\big(e^{r(T-\tau)}\,\min(L_\tau, D_\tau)\big)\big] \\
&= \int_{\ln(D_t/A_t)}^{\infty} u_L\big(L_T + \delta\big[\alpha A_t e^{y+g(T-t)} - L_T\big]^+ - \big[L_T - A_t e^{y+g(T-t)}\big]^+\big)\, g^{(1)}(y, t, T, A_t, D_t)\, dy \\
&\qquad + \int_t^T u_L\Big(e^{r(T-t)+(g-r)(\tau-t)}\,\min(L_t, D_t)\Big)\, f^{(1)}(t, \tau, A_t, D_t)\, d\tau\,. \qquad (23)
\end{aligned}$$

In the case of power utility (see Example 2.2), the latter integrals can be further simplified.

Proof of Theorem 3.2. Theorem 3.2 can rather straightforwardly be derived using the previous results. Note that the regulatory barrier $K_t = K_0 e^{gt}$ is always hit prior to default due to the continuity of the process A. Up to time $\hat{\tau}$ the process A is a geometric Brownian motion with strategy θ_1 allowing us to use the density $f^{(1)}(t, \hat{\tau}, A_t, K_t)$ from Lemma 6.1 for $\hat{\tau}$. At time $\hat{\tau}$, we are back in the situation that is already solved in Theorem 3.1: One has to adapt the initial values for A, D, and L. Furthermore, the time to maturity is now $T - \hat{\tau}$ instead of T and the investment strategy is now θ_2. If the regulatory threshold K is never hit, we can in analogy to the proof of Theorem 3.1 compute the expected utility of the terminal payoffs to get the first terms of $\zeta_L(A_t, D_t, K_t, L_t, T)$, respectively $\zeta_E(A_t, D_t, K_t, L_t, T)$:

$$\zeta_L(A_t, D_t, K_t, L_t, t, T) = \int_t^T \kappa_L^{(2)}(K_{\hat{\tau}}, D_{\hat{\tau}}, L_{\hat{\tau}}, \hat{\tau}, T) \cdot f^{(1)}(t, \hat{\tau}, A_t, K_t)\, d\hat{\tau}$$

$$+ \int_{\ln(K_t/A_t)}^{\infty} u_L\big(L_T + \delta\big[\alpha A_t e^{y+g(T-t)} - L_T\big]^+ - \big[L_T - A_t e^{y+g(T-t)}\big]^+\big)\, g^{(1)}(y, t, T, A_t, K_t)\, dy,$$

$$= \int_t^T \int_{\ln(D_t/K_t)}^{\infty} u_L\big(L_T + \delta\big[\alpha K_{\hat{\tau}} e^{y+g(T-\hat{\tau})} - L_T\big]^+ - \big[L_T - K_{\hat{\tau}} e^{y+g(T-\hat{\tau})}\big]^+\big)$$

$$\cdot f^{(1)}(t, \hat{\tau}, A_0, K_0) \cdot g^{(2)}(y, \hat{\tau}, T, K_{\hat{\tau}}, D_{\hat{\tau}})\ dy\, d\hat{\tau}$$

$$+ \int_t^T \int_{\hat{\tau}}^T u_L\Big(e^{r(T-t)+(g-r)(\hat{\tau}-t)} \min(L_t, D_t)\Big) \cdot f^{(1)}(t, \hat{\tau}, A_t, K_t) \cdot f^{(2)}(\hat{\tau}, \tau, K_{\hat{\tau}}, D_{\hat{\tau}})\ d\tau\, d\hat{\tau}$$

$$+ \int_{\ln(K_t/A_t)}^{\infty} u_L\big(L_T + \delta\big[\alpha A_t e^{y+g(T-t)} - L_T\big]^+ - \big[L_T - A_t e^{y+g(T-t)}\big]^+\big)\, g^{(1)}(y, t, T, A_t, K_t)\, dy$$

$$\zeta_E(A_t, D_t, K_t, L_t, t, T) = \int_t^T \kappa_E^{(2)}(K_{\hat{\tau}}, D_{\hat{\tau}}, L_{\hat{\tau}}, \hat{\tau}, T) \cdot f^{(1)}(t, \hat{\tau}, A_t, K_t)\ d\hat{\tau}$$

$$+ \int_{\ln(K_t/A_t)}^{\infty} \big(\big[A_t e^{y+g(T-t)} - L_T\big]^+ - \delta\big[\alpha A_t e^{y+g(T-t)} - L_T\big]^+\big)\, g^{(1)}(y, t, T, A_t, K_t)\, dy$$

$$= \int_t^T \int_{\ln(D_t/K_t)}^{\infty} \big(\big[K_{\hat{\tau}} e^{y+g(T-\hat{\tau})} - L_T\big]^+ - \delta\big[\alpha K_{\hat{\tau}} e^{y+g(T-\hat{\tau})} - L_T\big]^+\big)$$

$$\cdot f^{(1)}(t, \hat{\tau}, A_t, K_t) \cdot g^{(2)}(y, \hat{\tau}, T, K_{\hat{\tau}}, D_{\hat{\tau}})\ dy\, d\hat{\tau}$$

$$+ \int_t^T \int_{\hat{\tau}}^T e^{r(T-t)+(g-r)(\hat{\tau}-t)} \max(D_t - L_t, 0)$$

$$\cdot f^{(1)}(t, \hat{\tau}, A_t, K_t) \cdot f^{(2)}(\hat{\tau}, \tau, K_{\hat{\tau}}, D_{\hat{\tau}})\ d\tau\, d\hat{\tau}$$

$$+ \int_{\ln(K_t/A_t)}^{\infty} \big(\big[A_t e^{y+g(T-t)} - L_T\big]^+ - \delta\big[\alpha A_t e^{y+g(T-t)} - L_T\big]^+\big)\, g^{(1)}(y, t, T, A_t, K_t)\, dy\,,$$

with $\kappa_L^{(i)}(\,\cdot\,)$, $\kappa_E^{(i)}(\,\cdot\,)$, f, and g as defined in Theorem 3.1. Again, power utility simplifies the given expressions.

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