Efficient Diversification under Generalized Almost Stochastic Dominance

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Abstract

Stochastic dominance (SD) has been identified as an important method for efficient diversification. However, the SD rule is too rigid in that it remains silent on some obvious preferences between two distributions for most investors as pointed out by Leshno and Levy (2002). Thus the purpose of this paper is to derive an efficient frontier according to generalized almost stochastic dominance (GASD) rules proposed by Tsetlin et al. (2015), which can effectively delete the choices which are not preferred by most economically important decision makers. We first respectively propose tests for portfolio admissibility and portfolio optimality under generalized almost first-degree stochastic dominance. We then propose tests for efficient diversification under generalized almost second-degree stochastic dominance. In each test, we demonstrate how to use computational and tractable linear programming to implement the tests and provide their applications in the stock markets.

Keywords: almost stochastic dominance, portfolio efficiency, mathematical programming

1 Introduction

The well-established theory of stochastic dominance (SD)[1] is an important criterion for understanding decision making under uncertainty. This criterion has been referred to as a powerful rule for efficient diversification, which is one of the essential decisions in finance. SD rules have two merits. First, the theory does not require strong assumptions on investor preferences and asset return distributions such as the influential mean-variance approach established by Markowitz (1952). Second, the nonparametric analysis allows the data to speak for themselves as pointed out by Post (2003).[2]

To derive tractable tests for SD efficiency, Bawa et al. (1985) utilize the convex stochastic dominance condition constructed by Fishburn (1974) to identify the members of optimal and dominated sets when more than two choice alternatives are compared. Post (2003) derives necessary and sufficient tests for the second-degree SD (SSD) efficiency by considering full diversification across the choice alternatives. He also suggests that these tests could be generalized to third-degree SD efficiency. Parallel to Post (2003), Kuosmanen (2004) establishes tests for first-degree SD (FSD) and SSD measures to identify an efficient set with full diversification.

Although SD rules have been demonstrated to be useful in testing for efficient diversification, they are too rigid for most investors as pointed out by Leshno and Levy (2002). In several cases, most investors could have an obvious preference between two distributions whereas SD rules remain silent on it. For instance, assume that there are two prospects X and Y, where X yields 0 with a probability of 10^{-6} and a million dollars with a probability of $1 - 10^{-6}$ and Y yields one dollar with certainty. Neither X nor Y dominates the other based on the SD rules. However, it is obvious that most investors would prefer X to Y. To overcome the above drawback of SD, Leshno and Levy (2002) propose "almost stochastic dominance (ASD)" as the decision criterion for most decision makers who are economically important. Using this rule, it can be shown that X is better than Y for most decision makers.

Similar to SD, ASD also involves a group of different criteria, corresponding to different

[1] The theory of SD is developed by Hadar and Russell (1969), Hanoch and Levy (1969), Rothschild and Stiglitz (1970), and Whitmore (1970).

[2] Several alternative frameworks have been proposed in the literature. For example, Arditti (1967) and Jean (1973) extend two-moment analysis to higher moments; Harlow (1991) and Campbell et al. (2001) consider downside risk in asset allocation; for some other issues related to models of risk-averse preferences, please refer to Ogryczak and Ruszczynski (1999) and Dentcheva and Ruszczynski (2006).

sets of preference assumptions. Leshno and Levy (2002) showed that almost first-degree SD (AFSD) is the ranking criterion for most insatiable decision makers who are economically important. Following Leshno and Levy (2002), Tzeng et al. (2013) redefined almost second-degree SD (ASSD) and showed it to be the ranking criterion for most risk-averse decision makers by excluding pathological preferences. Furthermore, Tsetlin et al. (2015) proposed generalized ASD (GASD) to integrate the results of both Leshno and Levy (2002) and Tzeng et al. (2013).

The purpose of this paper is to derive the efficient frontier according to the GASD rules proposed by Tsetlin et al. (2015). We first derive tests for generalized almost FSD (GAFSD)[3] efficiency which include tests for FSD efficiency as special cases. There are two types of FSD efficiency tests in the literature. One is proposed by Kuosmanen (2004) who developed the test according to the distribution condition of FSD. Kopa and Post (2009) emphasize that the efficient portfolio set in Kuosmanen (2004) is only admissible. Some elements in the set are not optimal choices. Thus, they propose another testing procedure to eliminate the non-optimal portfolios. In this paper, we respectively propose a GAFSD admissibility test and a GAFSD optimality test.

We then propose tests for efficient diversification for generalized almost SSD (GASSD). Our GASSD efficiency test includes the SSD efficiency test proposed by Post (2003) as a special case. Note that Kopa and Post (2009) argued that SSD admissibility and SSD optimality are equivalent in a portfolio context. Thus, we directly develop a necessary and sufficient test statistic for GASSD efficiency.

To show the merits of efficient diversification under GASD, we demonstrate the use of linear programming to implement the GAFSD admissibility test, GAFSD optimality test and GASSD efficiency test, respectively. The linear programming methods we propose are computational and tractable. Furthermore, we employ a numerical example to derive efficient portfolios by using FSD and SSD efficiency tests as well as GAFSD and GASSD efficiency tests. Our numerical examples show that our approaches could effectively shrink the size of the efficient sets.

Furthermore, ASD has been applied to several issues in the finance literature.[4] We then

[3]GAFSD proposed by Tsetlin et al. (2015) is the same as AFSD defined by Leshno and Levy (2002).

[4]For example, Bali et al. (2009) show that ASD rules can support the popular investment practice suggesting a higher stock to bond ratio for long investment horizons, where this common practice cannot be

provide applications of our methodologies in a practical case where investors could form a three-asset portfolio among bonds, stocks and hedge funds. Our empirical application is close to, but differs from, Bali et al. (2013) in the following ways. First, Bali et al. (2013) adopt the alleged concept of ASSD proposed by Leshno and Levy (2002), which was further corrected by Tzeng et al. (2013). Our paper employs GASSD established by Tsetlin et al. (2015), which includes Tzeng et al. (2013) as a special case. Second, Bali et al. (2013) find that several types of hedge funds dominate the U.S. equity and bond markets. Our methodologies provide a further test on whether there exists a portfolio which dominates a 100% holding of hedge funds.

We find that, no matter which types of hedge funds are considered, a 100% hedge fund portfolio satisfies GAFSD admissibility, GAFSD optimality, as well as GASSD efficiency. In general, we also find that a 100% stock portfolio satisfies GAFSD admissibility and optimality but is not GASSD efficient. Furthermore, a 100% bond portfolio is FSD and SSD efficient but not GASD efficient in most cases.

The remainder of this paper is organized as follows. Section 2 reviews the definition of GASD. Section 3 first introduces two types of GAFSD efficiency tests, including the GAFSD admissibility test and GAFSD optimality test. Then, the GASSD efficiency test is also proposed. Section 4 illustrates our tests by means of numerical examples. Section 5 shows the empirical results from the hedge fund data. Finally, Section 6 briefly concludes the paper.

2 Generalized Almost Stochastic Dominance

In this section, we review the definitions for GAFSD and GASSD provided in Tsetlin et al. (2015). Let u denote an investor's utility function. Let u' and u'' denote the first and the second derivatives of u. Define the following utility classes:

$$U_1(\varepsilon_1) = \left\{ u \,\middle|\, u' > 0 \text{ and } \sup\{u'\} \leq \inf\{u'\}\left(\frac{1}{\varepsilon_1} - 1\right) \right\}, \tag{1}$$

explained by other well-known decision rules such as mean-variance or SD rules. Bali et al. (2013) show that the U.S. equity market and the U.S. Treasury market are almost stochastically dominated by different types of hedge funds. Based on the concept of ASD, Denuit et al. (2014) generate the rule of marginal conditional SD proposed by Yitzhaki and Olkin (1991) and Shalit and Yitzhaki (1994). They demonstrate how to construct an efficient frontier under the proposed almost marginal conditional SD criterion.

$$\text{and } U_2(\varepsilon_1, \varepsilon_2) = \left\{ u \;\middle|\; \begin{array}{c} u' > 0,\, u'' < 0, \\ \sup\{u'\} \leq \inf\{u'\}\left(\frac{1}{\varepsilon_1} - 1\right) \\ \sup\{-u''\} \leq \inf\{-u''\}\left(\frac{1}{\varepsilon_2} - 1\right). \end{array} \right\}, \tag{2}$$

where $\varepsilon_i \in \left[0, \frac{1}{2}\right]$, $i = 1, 2$. When ε_i decreases, $i = 1, 2$, the sets of $U_1(\varepsilon_1)$ and $U_2(\varepsilon_1, \varepsilon_2)$ become larger. When ε_1 approaches zero, $U_1(\varepsilon_1)$ expands to the set containing all insatiable preferences. If both ε_1 and ε_2 approach zero, then $U_2(\varepsilon_1, \varepsilon_2)$ becomes the set containing all insatiable and risk-averse preferences.

Let j and k be two risky portfolios, and F_j and F_k denote two cumulative distribution functions (CDFs) of portfolios j and k, respectively. The definitions of ε_1-GAFSD and $(\varepsilon_1, \varepsilon_2)$-GASSD are as follows:

Definition 1 *Portfolio j dominates portfolio k in terms of ε_1-GAFSD if $E_{F_j}(u) \geq E_{F_k}(u)$ for all $u \in U_1(\varepsilon_1)$.*

Definition 2 *Portfolio j dominates portfolio k in terms of $(\varepsilon_1, \varepsilon_2)$-GASSD if $E_{F_j}(u) \geq E_{F_k}(u)$ for all $u \in U_2(\varepsilon_1, \varepsilon_2)$.*

Similar to the SD rules, GASD also satisfies a hierarchy property: lower-degree dominance is stronger than higher-degree dominance. Thus, ε_1-GAFSD implies $(\varepsilon_1, \varepsilon_2)$-GASSD. Moreover, if portfolio j dominates portfolio k by FSD, it means that $E_{F_j}(u) \geq E_{F_k}(u)$ for all $u \in U_1(0)$. Since $U_1(\varepsilon_1) \subset U_1(0)$, Definition 1 concludes that portfolio j also dominates portfolio k by ε_1-GAFSD. Similarly, if portfolio j dominates portfolio k by SSD, then j also dominates portfolio k by $(\varepsilon_1, \varepsilon_2)$-GASSD. Thus, using the rules for ε_1-GAFSD and $(\varepsilon_1, \varepsilon_2)$-GASSD could eliminate more"inefficient" portfolios than the rules for FSD and SSD, respectively.

3 Linear Programming Tests

This section includes three parts: the ε_1-GAFSD admissibility test, the ε_1-GAFSD optimality test and the $(\varepsilon_1, \varepsilon_2)$-GASSD efficiency test. The ε_1-GAFSD admissibility test is the procedure used to identify the non-dominated portfolio set under the ε_1-GAFSD rule, whereas the ε_1-GAFSD optimality test is designed to eliminate the portfolios in the non-dominated set which cannot be supported by maximizing expected utility. The $(\varepsilon_1, \varepsilon_2)$-GASSD efficiency test shows how to identify the non-dominated portfolio set under the $(\varepsilon_1, \varepsilon_2)$-GASSD rule.

In the empirical portfolio analysis, the underlying distributions are estimated from real data, which are discrete. Thus, we consider a finite sample of N assets from T states of nature. The probability of each state is assumed to be $1/T$. Let $Y = (Y_1, Y_2, \ldots, Y_N)$ denote the matrix of N underlying assets, and each $Y_i = (Y_{i1}, Y_{i2}, \ldots, Y_{iT})'$ denotes a T states vector of asset i. To form the portfolio return, we define a column vector $\lambda \in \Lambda$ as portfolio weights, where $\Lambda \equiv \left\{\lambda \in \mathbb{R}^N \left| \Sigma_{i=1}^N \lambda_i = 1 \right.\right\}$ represents their feasible domain. All possible market portfolio sets are presented by $\psi \equiv \left\{y \in \mathbb{R}^T \left| y = Y\lambda, \lambda \in \Lambda \right.\right\}$.

Following the literature, the corresponding empirical distribution function is constructed as the CDFs in the tests. For a portfolio j, the elements of the portfolio return profile $y_j = (y_{j1}, y_{j2}, \ldots, y_{jT})$ are rearranged in a non-decreasing order. Let $x_j \equiv (x_{j1}, x_{j2}, \ldots, x_{jT})$ denote the resulting ranked vector (i.e., $x_{j1} \leq x_{j2} \leq \cdots \leq x_{jT}$) from the corresponding return vector y_j. We follow the literature and apply the GASD to the empirical distribution functions.

3.1 GAFSD Admissibility Test

To construct the ε_1-GAFSD admissibility test, we follow the approach in Kuosmanen (2004) but modify the dominance condition from FSD to ε_1-GAFSD. Let us first review the FSD admissibility test in Kuosmanen (2004). To prevent the information on the cross-sectional structure from being lost during the transformation from y to x for each asset, the establishment of the FSD admissibility test is based on the ranked evaluated portfolio y_0 and all possible market return profiles $Y^0\lambda^0$, where $Y^0 = (y_0, Y)$ represents all the feasible assets in the market, and λ^0 the vector of portfolio weights. Thus, instead of using x directly, he defines a permutation matrix P to rank the evaluated portfolio $y_0 \in \psi$ in different orders. Let $\{Py_0\}$ be the set of all permutations of y_0, where

$$P \in \Pi = \left\{ [P_{ts}]_{T\times T} \left| P_{ts} \in \{0,1\}, \sum_{s=1}^{T} P_{ts} = \sum_{t=1}^{T} P_{ts} = 1, \forall s, t \right. \right\}.$$

Thus, all permutations of y_0 can generate identical empirical distribution functions.

Kuosmanen (2004) further provides the following Theorem for the FSD admissibility test.

Theorem 1 ***FSD Admissibility Test (Kuosmanen, 2004)***

The portfolio return profile $y_0 \in \psi$ is empirically FSD efficient if and only if $\theta(y_0) = 0$, where $\theta(y_0)$ is solved from

$$\theta(y_0) = \max_{\lambda^0, P} \frac{1}{T} \left[\sum_{t=1}^{T} \sum_{i=1}^{N+1} Y_{it}^0 \lambda_i^0 - \sum_{t=1}^{T} y_{0t} \right] \tag{3}$$

$$\begin{aligned} s.t. \ \sum_{i=1}^{N+1} Y_{it}^0 \lambda_i^0 \ &\geq \ \sum_{s=1}^{T} P_{ts} y_{0s}, \ \forall t && (4) \\ P \ &\in \ \Pi && (5) \\ \lambda^0 \ &\in \ \Lambda && (6) \end{aligned}$$

The first constraint in Theorem 1 is the dominance condition. When the portfolio with weight λ^0 dominates the evaluated portfolio y_0 by FSD, the FSD rule requires that the CDF of the portfolio with weight λ^0 be smaller than that of y_0 for all states. Note that the distribution functions are monotonically increasing step functions, which have the step height $1/T$, and width $\sum_{i=1}^{N+1} Y_{it}^0 \lambda_i^0$ and y_{0t}. If the portfolio with weight λ^0 dominates y_0 by FSD, then it is equivalent to requiring that $\sum_{i=1}^{N+1} Y_{it}^0 \lambda_i^0 \geq y_{0t}$, $\forall t$. However, since all permutations of y_0 can generate identical empirical distribution functions, the condition can be rewritten as

$$\sum_{i=1}^{N+1} Y_{it}^0 \lambda_i^0 \geq \sum_{s=1}^{T} P_{ts} y_{0s}, \forall t.$$

$\theta(y_0)$ in Theorem 1 is the maximum value of the difference in the means between the portfolio with weight λ^0 and y_0. Theorem 1 indicates that y_0 being empirically FSD efficient in ψ means that it is not possible to find a portfolio in the market which has a greater mean than y_0 such that this portfolio is a dominating portfolio based on FSD.

To construct an ε_1-GAFSD admissibility test, the dominance constraint (4) needs to be modified. This test checks whether we can construct a portfolio which dominates the evaluated portfolio using the necessary and sufficient distribution conditions for ε_1-GAFSD in Tsetlin et al. (2015): Portfolio j dominates portfolio k by ε_1-GAFSD for $0 < \varepsilon_1 < \frac{1}{2}$ if and only if

$$\int_{S_1} [F_j(x) - F_k(x)]\, dx \le \varepsilon_1 \int_{\underline{x}}^{\overline{x}} |F_j(x) - F_k(x)|\, dx \tag{7}$$

where $S_1 = \{x \in [\underline{x}, \overline{x}]\, |F_j(x) > F_k(x)\}$.

Using the empirical distribution functions, the above necessary and sufficient condition (7) can be written as follows:

Theorem 2 *Portfolio j dominates portfolio k by ε_1-GAFSD for $0 < \varepsilon_1 < \frac{1}{2}$ if and only if*

$$\sum_{t \in \hat{S}_1} (x_{kt} - x_{jt}) \le \varepsilon_1 \sum_{t=1}^{T} |x_{kt} - x_{jt}|\,,\ where\, \hat{S}_1 = \{t\, |x_{jt} < x_{kt}\}\,. \tag{8}$$

Proof. The empirical distribution functions are monotonically increasing step functions, which have the step height $1/T$, and width x_{jt} and x_{kt}. Thus, the distance between $F_j(x)$ and $F_k(x)$ in Equation (7) can be obtained by comparing the values of x_{jt} and x_{kt}. ■

As in Kuosmanen (2004), we also use the permutations of vector y_0, Py_0. The ε_1-GAFSD dominating set of evaluated portfolio y_0 can be defined as follows:

Definition 3 *The ε_1-GAFSD denominating set of evaluated portfolio y_0 is*

$$\begin{aligned} D(y_0) &\equiv \left\{ y \in \mathbb{R}^T \left| \frac{\Sigma_{t \in \hat{S}_1} (Py_{0t} - y_t)}{\Sigma_{t=1}^{T} |Py_{0t} - y_t|} \le \varepsilon_1 \right. \right\} \\ &= \left\{ y \in \mathbb{R}^T \left| (1 - \varepsilon_1) \sum_{t \in \hat{S}_1} (Py_{0t} - y_t) + \varepsilon_1 \sum_{t \in \hat{S}'_1} (Py_{0t} - y_t) \le 0 \right. \right\}. \end{aligned} \tag{9}$$

where $\hat{S}_1 = \{t \in \mathbb{N}\, |Py_{0t} > y_t\}$*and* $\Omega = \hat{S}_1 \cup \hat{S}'_1$.

The concept of $D(y_0)$ is that there exists a P such that the sum of differences in $\hat{S}_1$ divided by the sum of the absolute differences in Ω is not greater than the violation ratio ε_1, and $\hat{S}_1$ represents the states of nature whereby the return of portfolio y is not greater than the evaluated portfolio y_0 at state t. Then, we are able to test the ε_1-GAFSD admissibility of any given portfolio return y_0 by checking whether there is any market portfolio y belonging to the ε_1-GAFSD dominating set. If there is no market portfolio y in the set of $D(y_0)$, then y_0 passes the ε_1-GAFSD admissibility test.

To establish an algorithm which can be solved by linear programming, we also consider all feasible assets in the market, $Y^0 = (y_0, Y)$. Thus, the dominating constraint in (9) is rewritten as

$$(1-\varepsilon_1)\sum_{t\in\hat{S}_1}\left(\sum_{s=1}^{T}P_{ts}y_{0s}-\sum_{i=1}^{N+1}Y_{it}^0\lambda_i^0\right)+\varepsilon_1\sum_{t\in\hat{S}_1'}\left(\sum_{s=1}^{T}P_{ts}y_{0s}-\sum_{i=1}^{N+1}Y_{it}^0\lambda_i^0\right)\leq 0. \tag{10}$$

Furthermore, we define

$$Z_t^- = \max\left\{-\left(\sum_{s=1}^{T}P_{ts}y_{0s}-\sum_{i=1}^{N+1}Y_{it}^0\lambda_i^0\right), 0\right\}\geq 0$$

and

$$Z_t^+ = \max\left\{\sum_{s=1}^{T}P_{ts}y_{0s}-\sum_{i=1}^{N+1}Y_{it}^0\lambda_i^0, 0\right\}\geq 0.$$

Thus, the programming model can be written as

$$\hat{\theta}(y_0) = \max_{\lambda^0, P, Z_t^+, Z_t^-} \quad \frac{1}{T}\left[\sum_{t=1}^{T}\sum_{i=1}^{N+1}Y_{it}^0\lambda_i^0-\sum_{t=1}^{T}y_{0t}\right]. \tag{11}$$

$$\begin{aligned}
s.t. \qquad\qquad Z_t^+ - Z_t^- \quad &= \quad \sum_{s=1}^{T}P_{ts}y_{0s}-\sum_{i=1}^{N+1}Y_{it}^0\lambda_i^0 && (12)\\
(1-\varepsilon_1)\sum_{t=1}^{T}Z_t^+ \quad &- \quad \varepsilon_1\sum_{t=1}^{T}Z_t^- \leq 0 && (13)\\
Z_t^+ \quad &\geq \quad 0\ \forall t && (14)\\
Z_t^- \quad &\geq \quad 0\ \forall t && (15)\\
P \quad &\in \quad \Pi && (16)\\
\lambda^0 \quad &\in \quad \Lambda && (17)
\end{aligned}$$

Under our setting, $\hat{\theta}(y_0)$ is solvable by a standard Mixed Integer Linear Programming (MILP) algorithm. Similar to the FSD efficient test in Kuosmanen (2004), the purpose of Problem (11) to (17) is to construct a portfolio, which is characterized by the weights λ^0, such that the constructed portfolio has the highest mean return, subject to the constraint that the portfolio dominates the evaluated portfolio y_0 in terms of ε_1-GAFSD. The permutation matrix P helps to take into account all possible orderings of states.

It is worth noting that if ε_1 approaches zero, i.e., the rule ε_1-GAFSD becomes FSD, then Problem (11) to (17) reduces to the FSD admissibility test in Kuosmanen (2004). It is because, in this case, the constraint (13) reduces to

$$\sum_{t=1}^{T} Z_t^+ \leq 0.$$

However, by constraint (14), $Z_t^+ \geq 0$ for all t. Therefore, it is shown that $Z_t^+ = 0$. From Equation (12), we obtain

$$\sum_{i=1}^{N+1} Y_{it}^0 \lambda_i^0 \geq \sum_{s=1}^{T} P_{ts} y_{0s} \ \forall t,$$

which consists of the FSD constraints (4) in Kuosmanen (2004).

The following theorem shows the necessary and sufficient measure of the ε_1-GAFSD admissibility test.

Theorem 3 ***ε_1-GAFSD Admissibility Test***
For $0 < \varepsilon_1 < 1/2$, the portfolio return profile y_0 is empirically ε_1-GAFSD efficient in ψ if and only if $\hat{\theta}(y_0) = 0$.

Proof. If the portfolio return profile y_0 is empirically ε_1-GAFSD efficient in ψ, then $D(y_0)$ only contains the elements constructed by y_0. Thus, $\hat{\theta}(y_0) = 0$. On the other hand, if $\hat{\theta}(y_0) = 0$, and it is assumed that there exists a $\hat{\lambda}^0$ and a $\hat{P}$ such that

$$(1-\varepsilon_1)\sum_{t\in\hat{S}_1}\left(\sum_{s=1}^{T}\hat{P}_{ts}y_{0s} - \sum_{i=1}^{N+1} Y_{it}^0\hat{\lambda}_i^0\right) + \varepsilon_1\sum_{t\in\hat{S}_1'}\left(\sum_{s=1}^{T}\hat{P}_{ts}y_{0s} - \sum_{i=1}^{N+1} Y_{it}^0\hat{\lambda}_i^0\right) < 0, \tag{18}$$

then by adding $\varepsilon_1 \sum_{t\in\hat{S}_1}\left(\sum_{s=1}^{T}\hat{P}_{ts}y_{0s}-\sum_{i=1}^{N+1}Y_{it}^0\hat{\lambda}_i^0\right)$ to both sides of the above constraint yields

$$\begin{aligned}
&\varepsilon_1 \sum_{t\in\hat{S}_1}\left(\sum_{s=1}^{T}\hat{P}_{ts}y_{0s}-\sum_{i=1}^{N+1}Y_{it}^0\hat{\lambda}_i^0\right)+(1-\varepsilon_1)\sum_{t\in\hat{S}_1}\left(\sum_{s=1}^{T}\hat{P}_{ts}y_{0s}-\sum_{i=1}^{N+1}Y_{it}^0\hat{\lambda}_i^0\right)\\
&+\varepsilon_1 \sum_{t\in\hat{S}_1'}\left(\sum_{s=1}^{T}\hat{P}_{ts}y_{0s}-\sum_{i=1}^{N+1}Y_{it}^0\hat{\lambda}_i^0\right)\\
<\ &\varepsilon_1 \sum_{t\in\hat{S}_1}\left(\sum_{s=1}^{T}\hat{P}_{ts}y_{0s}-\sum_{i=1}^{N+1}Y_{it}^0\hat{\lambda}_i^0\right).
\end{aligned}$$

Rearranging the above condition yields

$$\varepsilon_1 \sum_{t}\left(\sum_{s=1}^{T}\hat{P}_{ts}y_{0s}-\sum_{i=1}^{N+1}Y_{it}^0\hat{\lambda}_i^0\right)+(1-2\varepsilon_1)\sum_{t\in\hat{S}_1}\left(\sum_{s=1}^{T}\hat{P}_{ts}y_{0s}-\sum_{i=1}^{N+1}Y_{it}^0\hat{\lambda}_i^0\right)<0.$$

Since $0<\varepsilon_1<0.5$, and $\sum_{t\in\hat{S}_1}\left(\sum_{s=1}^{T}\hat{P}_{ts}y_{0s}-\sum_{i=1}^{N+1}Y_{it}^0\hat{\lambda}_i^0\right)>0$, we have

$$\sum_{t}\left(\sum_{s=1}^{T}\hat{P}_{ts}y_{0s}-\sum_{i=1}^{N+1}Y_{it}^0\hat{\lambda}_i^0\right)<0.$$

Since $\sum_t\sum_{s=1}^{T}\hat{P}_{ts}y_{0s}=\sum_{t=1}^{T}y_{0t}$, we have

$$\hat{\theta}(y_0)=\frac{1}{T}\left[\sum_{t=1}^{T}\sum_{i=1}^{N+1}Y_{it}^0\hat{\lambda}_i^0-\sum_{t=1}^{T}y_{0t}\right]>0,$$

which contradicts the assumption $\hat{\theta}(y_0)=0$. ■

This admissibility test not only examines whether the evaluated portfolio is efficient, but also suggests an alternative portfolio weight, λ^0, which dominates the evaluated portfolio y_0 in terms of ε_1-GAFSD.

3.2 GAFSD Optimality Test

According to Kopa and Post (2009), FSD optimality refers to a portfolio that achieves a higher expected utility than all other portfolios for certain investors with increasing utility functions. To extend the methodology to the ε_1-GAFSD optimality, we follow the concept of Kopa and Post (2009) to propose a representative utility function in utility class $U_1(\varepsilon_1)$ and then introduce the ε_1-GAFSD optimality test.

Let us first briefly review the test established by Kopa and Post (2009). To develop the test, Kopa and Post (2009) propose that any utility function can be transformed into a piecewise-constant function with increments only at x_{0t}, $t = 1, 2, ..., T$. Without loss of generality, they consider all piecewise-constant increasing utility functions to be in the following set:

$$U_1(y_0) = \left\{ u \,\middle|\, u(x_{01}) = 0,\ u(y) = \sum_{t=1}^{T} a_t I(y \geq x_{0t}),\ \sum_{t=1}^{T} a_t = 1, \text{ and } a_t \in R^+ \right\}, \tag{19}$$

where

$$I(y \geq x_{0t}) = \begin{cases} 1 \text{ for } y \geq x_{0t} \\ 0 \text{ otherwise} \end{cases}. \tag{20}$$

Kopa and Post (2009) define portfolio $y_0 \in \psi$ as being FSD optimal if it is the optimal solution to the expected utility function for at least some utility function $u \in U_1(y_0)$ such that

$$\sum_{t=1}^{T} u(y_{0t}) - \sum_{t=1}^{T} u(Y_t\lambda) \geq 0,\ \forall \lambda \in \Lambda. \tag{21}$$

Otherwise, y_0 is FSD non-optimal. Thus, they construct the equal notations of the test statistic:

$$\begin{aligned} \xi(y_0, Y\lambda) &= \frac{1}{T}\left[\min_{u \in U_1(y_0)} \max_{\lambda \in \Lambda} \sum_{t=1}^{T} u(Y_t\lambda) - \sum_{t=1}^{T} u(y_{0t})\right] \\ &= \frac{1}{T}\left[\min_{u \in U_1(y_0)} \max_{\lambda \in \Lambda} \sum_{t=1}^{T} \sum_{s=k(\lambda)}^{T} a_s \left(I(Y_t\lambda \geq x_{0s}) - I(y_{0t} \geq x_{0s})\right)\right] \\ &= \frac{1}{T}\left[\min_{u \in U_1(y_0)} \max_{\lambda \in \Lambda} \sum_{s=k(\lambda)}^{T} a_s \left(\bar{h}_s - h_{0s}\right)\right], \end{aligned} \tag{22}$$

where $k(\lambda)$ denotes the order of the second smallest return of the portfolio with weight λ. $\bar{h}_s = \sum_{t=1}^{T} I(Y_t\lambda \geq x_{0s})$ represents the number of the returns of portfolio λ exceeding the sth smallest return of x_{0s} and $h_{0s} = \sum_{t=1}^{T} I(y_{0t} \geq x_{0s})$ is the number of the returns of y_0 exceeding the sth smallest return of x_{0s}. The intuition behind this test measure is to check whether there exists a portfolio λ which generates a higher utility than the evaluated portfolio. They further provide the following theorem:

Theorem 4 ***FSD Optimality Test (Kopa and Post, 2009)***
Portfolio y_0 is FSD optimal if and only if $\delta^*(H_0) = 0$, where $\delta^*(H_0)$ is the solution of the following problem:

$$\begin{aligned} \delta^*(H_0) &= \min_{\gamma, a_s} \delta \\ s.t. & \sum_{s=k}^{T} a_s \left(\bar{h}_s - h_{0s}\right) \leq \delta \ \forall \bar{h} \in H_0 \end{aligned} \tag{23}$$

where H_0 is a set containing all possible numbers of $\bar{h}$. If $\delta^*(H_0) > 0$ for some H_0, then y_0 is FSD nonoptimal.

To modify the above test, it is worth noting that the ε_1-GAFSD optimality involves identifying a portfolio with higher expected utility in $U_1(\varepsilon_1)$ instead of in U_1 as in Kopa and Post (2009). Thus, we follow the tests in Kopa and Post (2009) but consider the following representative piecewise-constant utility function:

$$\begin{aligned} U_1(y_0, \varepsilon_1) = & \left\{ u \in U_1(\varepsilon_1) \middle| u(y) = \sum_{j=1}^{T} a_t I(y \geq x_{0t}), \sum_{j=1}^{T} a_s = 1, \right. \\ & \left. \frac{a_j}{x_{0j} - x_{0j-1}} \leq \inf \left\{ \frac{a_j}{x_{0j} - x_{0j-1}} \right\} \left(\frac{1}{\varepsilon_1} - 1 \right), \forall j = 1, \ldots, T \right\}. \end{aligned} \tag{24}$$

The ε_1-GAFSD optimal portfolio can be defined as follows.

Definition 4 *Portfolio $y_0 \in \psi$ is ε_1-GAFSD optimal if it is the optimal solution of the expected utility function for at least some utility function $u \in U_1(y_0, \varepsilon_1)$ such that*

$$\sum_{t=1}^{T} u(y_{0t}) - \sum_{t=1}^{T} u(Y_t \lambda) \geq 0, \ \forall \lambda \in \Lambda. \tag{25}$$

Otherwise, y_0 is ε_1-GAFSD non-optimal.

Thus, we modify the statistics $\delta^*(H_0)$ in Theorem 4 by adding a constraint on marginal utility and developing the following Theorem for the ε_1-GAFSD optimality test.

Theorem 5 ε_1-***GAFSD Optimality Test***

Portfolio y_0 is ε_1-GAFSD optimal if and only if $\hat{\delta}^*(H_0) = 0$, where $\hat{\delta}^*(H_0)$ is the solution of the following problem:

$$\begin{aligned} \hat{\delta}^*(H_0) &= \min_{\gamma, a_s} \delta \\ & s.t. \ \sum_{s=k}^{T} a_s \left(\bar{h}_s - h_{0s}\right) \leq \delta \ \forall \bar{h} \in H_0 \qquad (26) \\ \gamma &\leq \frac{a_t}{x_t - x_{t-1}} \leq \gamma \left(\frac{1}{\varepsilon_1} - 1\right), \ t = k, ..., T, \qquad (27) \end{aligned}$$

where H_0 is a set containing all possible numbers of $\bar{h}$. If $\hat{\delta}^*(H_0) > 0$ for some H_0, then y_0 is ε_1-GAFSD nonoptimal.

Proof. Note that $\gamma \leq \frac{a_t}{x_t - x_{t-1}}$ for $t = k, ..., T$ results in $\gamma = \inf\left\{\frac{a_t}{x_t - x_{t-1}}\right\}$. On the other hand, $\frac{a_t}{x_t - x_{t-1}} \leq \gamma\left(\frac{1}{\varepsilon_1} - 1\right)$ for $t = k, ..., T$ results in $\frac{a_t}{x_t - x_{t-1}} \leq \inf\left\{\frac{a_t}{x_t - x_{t-1}}\right\}\left(\frac{1}{\varepsilon_1} - 1\right)$ for $t = k, ..., T$. Thus, the second constraint

$$\gamma \leq \frac{a_t}{x_t - x_{t-1}} \leq \gamma\left(\frac{1}{\varepsilon_1} - 1\right), \ t = k, ..., T$$

indicates that the utility set is $U_1(y_0, \varepsilon_1)$ rather than $U_1(y_0)$. The proof is omitted since it is similar to the proof of Theorem 4. ■

3.3 GASSD Efficiency Test

To derive empirical tests for the $(\varepsilon_1, \varepsilon_2)$-GASSD efficiency of a given portfolio, we follow Post (2003) to construct representative piecewise-linear utility functions. Post (2003) derives tests for the SSD efficiency of a given portfolio with respect to all possible portfolios. He defines portfolio $y_0 \in \psi$ as being SSD inefficient if for all utility functions $u \in U_2 = \{u \,|u' > 0, u'' < 0\}$, i.e., the maximum expected utility is greater than the expected utility of Portfolio y_0, that is,

$$\min_{u \in U_2} \max_{\lambda \in \Lambda} \left\{\sum_{t=1}^{T} u(Y_t\lambda) - \sum_{t=1}^{T} u(y_{0t})\right\} > 0. \qquad (28)$$

Otherwise, y_0 is SSD efficient. He adopts the piecewise-linear utility functions in U_2:

$$p(x|\alpha,\beta) \equiv \min_{t\in\{1,...,T\}}(\alpha_t + \beta_t x), \tag{29}$$

where $\alpha \in R^T$ and $\beta \in R^{T+}$ with $\beta_1 \geq \beta_2 \geq ... \geq \beta_T = 1$ are vectors of the T states intercept coefficients and the T states slope coefficients, respectively. He further presents the following theorem for the SSD efficiency tests:

Theorem 6 ***SSD Efficiency Test (Post, 2003)***
Portfolio y_0 is SSD efficient if and only if $\phi^*(A_0) = 0$, where $\phi^*(A_0)$ is the solution of the following problem:

$$\phi^*(A_0) = \min_{\phi,\beta}\ \phi$$

$$s.t.\quad \frac{1}{T}\sum_{t=1}^{T}\beta_t\left(x_t^T - x_{it}\right) + \phi \geq 0 \quad i = 1,2,...,N \tag{30}$$

$$\beta_{t-1} - \beta_t \geq 0 \quad t = 1,2,...,T \tag{31}$$

$$\beta_T = 1. \tag{32}$$

A_0 is an $N \times T$ matrix containing all possible portfolio payoffs. Alternatively, portfolio y_0 is SSD inefficient if and only if $\phi^*(A_0) > 0$.

Based on the theorem of $(\varepsilon_1,\varepsilon_2)$-GASSD, we provide the following definition of $(\varepsilon_1,\varepsilon_2)$-GASSD efficiency:

Definition 5 *Portfolio $y_0 \in \psi$ is $(\varepsilon_1,\varepsilon_2)$-GASSD inefficient if and only if for all utility functions, $u \in U_2(\varepsilon_1,\varepsilon_2)$, the maximum expected utility is greater than the expected utility of Portfolio y_0, that is,*

$$\min_{u\in U_2^*}\max_{\lambda\in\Lambda}\left\{\sum_{t=1}^{T}u(Y_t\lambda) - \sum_{t=1}^{T}u(y_{0t})\right\} > 0 \tag{33}$$

Otherwise, y_0 is $(\varepsilon_1,\varepsilon_2)$-GASSD efficient.

Thus, to derive the $(\varepsilon_1,\varepsilon_2)$-GASSD efficiency test, we further modify the statistics $\phi^*(A_0)$ in Theorem 6 by adding in the constraints on β_t and present the following test:

Theorem 7 $(\varepsilon_1, \varepsilon_2)$-*GASSD* ***Efficiency Test***

Portfolio y_0 is $(\varepsilon_1, \varepsilon_2)$-GASSD efficient if and only if $\hat{\phi}^*(A_0) = 0$, where $\hat{\phi}^*(A_0)$ is the solution of the following problem:

$$\begin{aligned} \hat{\phi}^*(A_0) &= \min_{\phi,\beta} \ \phi \\ s.t. \quad \frac{1}{T}\sum_{t=1}^{T} \beta_t \left(x_t^T - x_{it}\right) + \phi &\geq 0 \quad i = 1, 2, ..., N && (34) \\ \beta_{t-1} - \beta_t &\geq 0 \quad t = 1, 2, ..., T && (35) \\ \beta_T &= 1 && (36) \\ \beta_T \left(\frac{1}{\varepsilon_1} - 1\right) &\geq \beta_1 && (37) \\ \frac{(\beta_{t-1} - \beta_t)}{(x_t - x_{t-1})} &\geq \gamma \quad t = k, ..., T && (38) \\ \gamma \left(\frac{1}{\varepsilon_2} - 1\right) &\geq \frac{(\beta_{t-1} - \beta_t)}{(x_t - x_{t-1})} \quad t = k, ..., T && (39) \end{aligned}$$

A_0 is an $N \times T$ matrix containing all possible portfolio payoffs. Alternatively, portfolio y_0 is $(\varepsilon_1, \varepsilon_2)$-GASSD inefficient if and only if $\phi^*(A_0) > 0$.

Proof. Since $u'' < 0$, $\sup\{u'\} = \beta_1$ and $\inf\{u'\} = \beta_T$. Thus, the third constraint

$$\beta_T \left(\frac{1}{\varepsilon_1} - 1\right) \geq \beta_1$$

is added to represent the condition $\sup\{u'\} \leq \inf\{u'\}\left(\frac{1}{\varepsilon_1} - 1\right)$. Furthermore, $\frac{(\beta_{t-1}-\beta_t)}{(x_t-x_{t-1})} \geq \gamma$ for $t = k, ..., T$ results in $\gamma = \inf\{-u''\}$. On the other hand, $\gamma\left(\frac{1}{\varepsilon_2} - 1\right) \geq \frac{(\beta_{t-1}-\beta_t)}{(x_t-x_{t-1})}$ for $t = k, ..., T$ results in $\frac{(\beta_{t-1}-\beta_t)}{(x_t-x_{t-1})} \leq \inf\{-u''\}\left(\frac{1}{\varepsilon_2} - 1\right)$ for $t = k, ..., T$. Thus, the constraint

$$\gamma \leq \frac{a_t}{x_t - x_{t-1}} \leq \gamma \left(\frac{1}{\varepsilon_1} - 1\right), \ t = k, ..., T$$

indicates that the utility set is $U_2(\varepsilon_1, \varepsilon_2)$. The proof is omitted since it is similar to the proof of Theorem 6. ∎

4 Numerical illustrations

In this section, we adopt a numerical example to demonstrate that our tests can efficiently eliminate portfolios which are viewed as dominated portfolios by most investors. The example is originally shown in Kopa and Post (2009) with five states ($T = 5$) of nature. Assume that there are three assets: (Y_1, Y_2, Y_3). The returns of these three assets in these five states are shown in Table 1. It is obvious that no individual asset (Y_1, Y_2, and Y_3) dominates any other by FSD. Thus, we form an evaluated portfolio $y_0 = \lambda_1 Y_1 + \lambda_2 Y_2 + \lambda_3 Y_3$ and use a grid with step size 0.01 for the portfolio weights.

[Insert Table 1 here]

The efficient portfolio weights for the GAFSD tests are shown in Figure 1. In each figure, the x-axis denotes the weights of the alternative Y_1, λ_1, and the y-axis denotes the weights of the alternative Y_2, λ_2. The lower triangle area is all possible portfolios of these three assets under the constraint, $\lambda_1 + \lambda_2 + \lambda_3 = 1$. The grey dots are portfolios that passed the admissibility test or optimality test, while the remaining portfolios failed the test and are classified as non-admissible or non-optimal. Panel A in Figure 1 shows the case where $\varepsilon_1 = 0$, i.e., the FSD efficiency test in Kuosmanen (2004) and the FSD optimality test Kopa and Post (2009). There are $1,127$ FSD admissible portfolios and 746 FSD optimal portfolios. Panel B, Panel C and Panel D respectively show that by allowing $\varepsilon_1 = 0.01, \varepsilon_1 = 0.03$ and $\varepsilon_1 = 0.05$,[5] more portfolios could be recognized as inefficient portfolios in our GAFSD admissibility test. In the optimality test, we successfully exclude many portfolios that are only chosen by investors with extreme utility functions.

[Insert Figure 1 here]

The results of the GASSD efficiency test are shown in Figures 2, 3 and 4. In Figure 2, Panel A, we set $\varepsilon_1 = 0$ and $\varepsilon_2 = 0$, i.e., the grey dots represent the SSD efficient portfolios. The figure shows that only 116 portfolios are left. From this case, all the risk-averse investors will prefer the portfolio with a lower weight in Y_1 because of the higher volatility in the Y_1

[5] Levy et al. (2010) used experimental data to estimate the critical parameter ε_1. According to their experiments, the inferior-allowed ε_1 for all insatiable subjects is 5.9%. Huang et al. (2014) further construct experiments according to almost Nth degree risk proposed by Tsetlin et al. (2015) and suggest that the inferior of the estimated allowed ε_1 for all insatiable subjects is less than 5.27%.

asset. Panels B, C and D in Figure 2 show the GASSD efficient portfolio when $\varepsilon_1 = 0$ and ε_2 is respectively equal to $0.01, 0.03$ and 0.05.[6] The results show that 79 portfolios will be eliminated as $\varepsilon_2 = 0.01$ and only 32 and 29 portfolios will be left when ε_2 is 0.03 and 0.05.

[Insert Figure 2 here]

In Figure 3, we consider the case where $\varepsilon_1 = 0.01$ and then obtain similar results to those shown in Figure 2. Most risk-averse investors prefer to attach less weight to the Y_1 asset and $\varepsilon_2 > 0$ helps investors to erase some non-efficient portfolios. After we allow $\varepsilon_1 = 0.03$, Figure 4 shows that we have 41 portfolios left, even though $\varepsilon_2 = 0$ for the risk-averse investors. When $\varepsilon_2 > 0$, we have 37, 31, and 28 portfolios left, corresponding to $\varepsilon_2 = 0.01, 0.03$ and 0.05.

[Insert Figure 3 here]

[Insert Figure 4 here]

5 Empirical analysis

In this section, we demonstrate the application of our models to real cases by analyzing whether a 100% hedge fund portfolio constitutes an efficient investment relative to all possible portfolios constructed from hedge funds, the U.S. equity index and the one-year U.S. Treasury bond for most insatiable and risk-averse investors. In the last 20 years, hedge funds have become one of the key investment vehicles and sources of capital in the world. Their option-based performance, active trading and diversified strategies have attracted a large amount of money from global investors and also the attention of academia. There are several studies that document the ability of hedge funds to deliver abnormal returns (Cao et al. 2013; Chen and Liang 2007) and also many that probe into the causes of their superior performance (Bali et al. 2011 and 2012). Specifically, Bali et al. (2013)[7] show that the U.S. equity market and the U.S. Treasury market are almost stochastically dominated by different types of hedge funds. However, the results of Bali et al. (2013) cannot guarantee that there does not exist

[6] According to the experimental results of Huang et al. (2014), the inferior of the allowed ε_2 for all risk-averse subjects is 2.20%, and 95% of the risk-averse subjects have an ε_2 greater than 5.74%.

[7] Although the findings in Bali et al. (2013) on the basis of AFSD are sound, their findings on the basis of ASSD need to be double-checked. This is because they adopted the alleged definition of ASSD, which has been corrected by Tzeng et al. (2013).

a diversified stock/bond/hedge fund portfolio which dominates 100% hedge fund portfolios for most investors.[8] This section sheds light on this issue.

We proceed with tests to analyze GAFSD admissibility and optimality and GASSD efficiency of the portfolio composed of hedge funds and the U.S. equity and bond markets.[9] We collect data from the Hedge Fund Research database and refer to the literature to eliminate survivorship bias, back-fill bias and multi-period sampling bias. The initial hedge funds contain a total of $11,867$ defunct funds and $6,853$ live funds over the period from January 1994 to December 2011. After the screening procedure, we leave $12,816$ hedge funds in our sample including $7,443$ dead funds and $5,373$ live funds. Then, we follow Denuit et al. (2014) and group funds into seven broad investment categories: Emerging Markets, Equity Market Neutral, Event Driven, Fund of Funds, Macro, Relative Value and Equity Hedge. The performance of the U.S. equity market and the performance of short-term U.S. Treasury securities are represented by the S&P500 index returns and the 1-year Treasury Bond returns, respectively.

Table 2 reports the summary statistics of the portfolios over the entire study period under a semi-annual base. Four out of seven hedge fund investment styles have higher average returns and lower standard deviations than the S&P 500 index. Except for the Macro hedge fund, the high-order moments and the Jarque-Bera (JB) statistics of the other hedge funds indicate significant departures from normality.

[Insert Table 2 here]

The left-hand side of Table 3 illustrates the admissibility and optimality classifications according to Theorem 3 and Theorem 5. The data are based on 36 semi-annual observations. We apply our tests to seven portfolios formed on one type of hedge fund, the U.S. equity and bond type. In the middle of Table 3, we respectively report whether a 100% hedge fund, 100% stock and 100% bond portfolio is ε_1-GAFSD efficient or optimal, where ε_1 is set as 0, 0.05 and 0.1. Note that the GAFSD tests reduce to FSD tests when $\varepsilon_1 = 0$. The right-hand side of Table 3 shows whether a 100% hedge fund, 100% stock or 100% bond portfolio is

[8] Denuit et al. (2014) show that the 100% hedge fund portfolio is efficient via their proposed almost marginal conditional SD criterion. The criterion gives the conditions under which most risk-averse investors prefer to increase the share of one risky asset over another in a given portfolio.

[9] Since the purpose of this section is to demonstrate the application, we only consider positive portfolio weights on each asset for simplicity.

$(\varepsilon_1, \varepsilon_2)$-GASSD efficient where $\varepsilon_1 = 0, 0.05, 0.1$, and $\varepsilon_2 = 0, 0.05, 0.1$ on the basis of Theorem 7. Also note that the GASSD test reduces to the SSD test when $\varepsilon_1 = \varepsilon_2 = 0$.

The results shown in Table 3 are consistent with the theoretical prediction that GAFSD non-admissibility implies GAFSD non-optimality. In each case, if the evaluated portfolio is not admissible then it is not optimal. Furthermore, te empirical results are also consistent with the theoretical prediction generated by Tsetlin et al. (2015) that GAFSD implies GASSD. If the evaluated portfolio is not admissible for a given ε_1, then it is not GASSD efficient in cases with the same ε_1 and for all ε_2.

[Insert Table 3 here]

Moreover, Table 3 shows that no matter which type of hedge fund is examined, a 100% hedge fund portfolio is not only GAFSD efficient but also GASSD efficient under the designed parameters. These results complement the findings in Bali et al. (2013), who find that most hedge funds dominate the U.S. equity market and bond market. We further demonstrate that, within the designed ε_1 and ε_2, no portfolios constructed from hedge funds, the U.S. equity index and the one-year U.S. Treasury bond can dominate a 100% hedge fund portfolio in terms of GAFSD and GASSD.

As for the S&P 500 index, we find that a 100% stock portfolio satisfies GAFSD admissibility and optimality except in the cases where emerging market hedge funds are included. If the portfolio may contain emerging market hedge funds, then a 100% stock portfolio is an efficient portfolio only under FSD criteria. When $\varepsilon_1 = 0.05$, our GAFSD admissibility test identifies that a 100% stock portfolio is dominated by a portfolio with investment weights of 66.3%, 27.5% and 6.2% attached to emerging market hedge funds, the S&P 500 and T-Bonds, respectively. The dominating portfolio is shown in Figure 5. Furthermore, we find that a 100% stock portfolio is not GASSD efficient except in the cases where the Fund of Funds category is included in the portfolio.

In addition, Table 3 indicates that a 100% bond portfolio is FSD admissible and optimal and SSD efficient. In the GAFSD admissibility tests, the 100% bond portfolio is efficient only in the Fund of Funds category. As $\varepsilon_1 = 0$ or 0.05 and $\varepsilon_2 = 0$, the 100% bond portfolio is an efficient portfolio when in the Emerging Markets, Event Driven, Fund of Funds and Equity Hedge categories, but it is not efficient if $\varepsilon_2 > 0$.

[Insert Figure 5 here]

6 Conclusion

The development of efficient investment sets is a classic issue in portfolio theory. In this paper, we have established new tests to obtain efficient diversified portfolios by using GASD criteria and establishing the algorithms of the GAFSD efficiency and optimality tests and the GASSD efficiency test for portfolio allocation. The FSD admissibility test, FSD optimality test and SSD efficiency test respectively proposed by Kuosmanen (2004), Kopa and Post (2009) and Post (2003) are special cases of our tests. All of our tests can be computed using linear programming.

As illustrated in our paper, the numerical results have shown that the set of efficient portfolios significantly decreases under the GASD criteria. The empirical applications have demonstrated that a 100% hedge fund portfolio satisfies admissibility and optimality in terms of GAFSD and is also GASSD efficient compared to all possible portfolios constructed from hedge funds, the U.S. equity index and the one-year U.S. Treasury bond. In general, we have also found that a 100% stock portfolio satisfies GAFSD admissibility and optimality but is not GASSD efficient, whereas a 100% bond portfolio is FSD and SSD efficient but not GASD efficient in most cases.

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Table 1: The Example of Kopa and Post (2009) with five states ($T = 5$).
Evaluated portfolio: $y_0 = \lambda_1 Y_1 + \lambda_2 Y_2 + \lambda_3 Y_3$

t	Y_1	Y_2	Y_3
1	-1	6	-4
2	-2	5.9	2
3	3.5	2.2	3
4	8.7	2	5
5	10	7	7.5

[]

Figure 1: Panel A. $\varepsilon_1 = 0, u \in U_1$
[b]

Figure 2: Panel B. $\varepsilon_1 = 0.01, u \in U_1(0.01)$

Figure 3: GAFSD Efficiency Tests.

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Figure 3: Panel C. $\varepsilon_1 = 0.03, u \in U_1(0.03)$
[b]

Figure 4: Panel D. $\varepsilon_1 = 0.05, u \in U_1(0.05)$

Figure 5: (continued) GAFSD Efficiency Tests.

Figure 6: $(0, \varepsilon_2 s)$-GASSD Efficiency Tests.

Figure 7: $(0.01, \varepsilon_2 s)$-GASSD Efficiency Tests.

Figure 8: $(0.03, \varepsilon_2 s)$-GASSD Efficiency Tests.

Figure 9: The CDFs of the 100% stock portfolio and the GAFSD dominating diversified portfolio, which has weights 66.3%, 27.5% and 6.2% in hedge funds, S&P 500 and Bond, respectively.

Table 2: Descriptive Statistics

Asset	Mean	Median	Std Dev	Skewness	Kurtosis	Min	Max	JB	p-value
Emerging Market	6.49%	8.75%	15.02%	-0.7422	3.8984	-35.78%	36.31%	4.51	0.0518
Macro	5.06%	5.51%	3.97%	-0.1040	2.2545	-2.59%	13.23%	0.89	0.5
Equity Hedge	5.32%	6.31%	7.93%	-1.0913	5.8446	-22.84%	21.29%	19.28	0.0035
Event Driven	5.01%	6.96%	6.81%	-1.8584	8.1601	-22.14%	14.38%	60.66	< 0.001
Relative Value	4.13%	4.90%	5.29%	-2.3448	11.6937	-19.17%	12.89%	146.36	< 0.001
Fund of Funds	0.50%	0.62%	1.66%	-0.6570	3.2184	-6.43%	5.95%	102.75	< 0.001
Equity Neutral	3.53%	3.40%	2.92%	-0.3146	4.2622	-5.42%	9.59%	2.98	0.0960
S&P500	3.38%	4.92%	11.21%	-0.8629	4.1519	-32.49%	20.87%	6.45	0.0296
1-year T-Bond	1.94%	1.91%	1.29%	0.1385	1.9796	0.02%	4.81%	1.68	0.2555

This table presents the descriptive statistics of semiannual returns on the hedge fund portfolios, S&P500 index, 1-year Treasury Bond for the sample period January 1994 to December 2011. We compute the equal-weighted average monthly returns of funds for each of the 7 investment styles. Jarque-Bera statistic, $JB = n[(S^2/6) + (K-3)^2/24]$, is a formal statistic for testing whether the returns are normally distributed, where n denotes the number of observations, S is skewness and K is kurtosis. JB follows Chi-square distribution with two degrees of freedom. The last column reports the corresponding p-values.

Table 3: GASD Efficiency tests

			Admissibility test			Optimality test				GASSD Efficiency test								
(l)4-6 (l)7-911-19 Hedge fund	100% holding	ε_1	0	0.05	0.1	0	0.05	0.1	ε_1 / ε_2	0 / 0	0 / 0.05	0 / 0.1	0.05 / 0	0.05 / 0.05	0.05 / 0.1	0.1 / 0	0.1 / 0.05	0.1 / 0.1
32cm Event Driven	fund		Y	Y	Y	Y	Y	Y		Y	Y	Y	Y	Y	Y	Y	Y	Y
	S&P 500		Y	Y	Y	Y	Y	Y		N	N	N	N	N	N	N	N	N
	TB		Y	Y	N	Y	N	N		Y	N	N	Y	N	N	N	N	N
32cm Equity Hedge	fund		Y	Y	Y	Y	Y	Y		Y	Y	Y	Y	Y	Y	Y	Y	Y
	S&P 500		Y	Y	Y	Y	Y	Y		N	N	N	N	N	N	N	N	N
	TB		Y	Y	N	Y	N	N		Y	N	N	Y	N	N	N	N	N
32cm Macro	fund		Y	Y	Y	Y	Y	Y		Y	Y	Y	Y	Y	Y	Y	Y	Y
	S&P 500		Y	Y	Y	Y	Y	Y		N	N	N	N	N	N	N	N	N
	TB		Y	N	N	Y	N	N		Y	N	N	N	N	N	N	N	N
32cm Relative Value	fund		Y	Y	Y	Y	Y	Y		Y	Y	Y	Y	Y	Y	Y	Y	Y
	S&P 500		Y	Y	Y	Y	Y	Y		N	N	N	N	N	N	N	N	N
	TB		Y	N	N	Y	N	N		Y	N	N	N	N	N	N	N	N
32cm FOF	fund		Y	Y	Y	Y	Y	Y		Y	Y	Y	Y	Y	Y	Y	Y	Y
	S&P 500		Y	Y	Y	Y	Y	Y		Y	Y	Y	Y	Y	Y	Y	Y	Y
	TB		Y	Y	Y	Y	N	N		Y	N	N	Y	N	N	N	N	N
32cm Emerging market	fund		Y	Y	Y	Y	Y	Y		Y	Y	Y	Y	Y	Y	Y	Y	Y
	S&P 500		Y	N	N	N	N	N		N	N	N	N	N	N	N	N	N
	TB		Y	Y	N	Y	N	N		Y	N	N	Y	N	N	N	N	N
32cmMarket Neutral	fund		Y	Y	Y	Y	Y	Y		Y	Y	Y	Y	Y	Y	Y	Y	Y
	S&P 500		Y	Y	Y	Y	Y	Y		N	N	N	N	N	N	N	N	N
	TB		Y	N	N	Y	N	N		Y	N	N	N	N	N	N	N	N

This table tests portfolios efficiency according to the test measures in Theorem **??**, Theorem **??**, and Theorem **??**. "Y" represents that the evaluated portfolio in the left-hand side category is efficient portfolio in comparison with all possible portfolios fromed by the evaluated asset and the other two feasible assets. On the contrary, "N" represents that the evaluated portfolio is inefficient.